Diophantine Equations · Lecture 16 · `alg-diophantine`

📼 **Course materials — Lecture 16:** lecture video · handout PDF (Lecture 16) · board notes (16.jpg) · homework answer key (Chapter 16) — all in the `amc8/video` folder. · 🖨️ **Printable PDF** (answers & solutions at the back)

🔑 Key facts

A **Diophantine equation** has more unknowns than equations — like $3x + y = 28$ — but demands *integer* (often positive-integer) solutions. Integrality is the missing equation: it shrinks "infinitely many solutions" down to a handful you can list.

For $ax + by = c$, once you have one solution, all the others **step by the opposite coefficients**: x changes by b while y changes by $-a$. Counting solutions = counting the steps that stay inside the allowed range.

Partner facts for the LCM/GCF half of the lecture: $\gcd(A, B) \cdot \operatorname{lcm}(A, B) = A \cdot B$, and in prime factorizations GCF takes the *smaller* exponent of each prime, LCM the *larger*.

🧰 Instructor's toolbox (from the lecture video & board notes)

> **1 · Name the unknowns and write the equation — even if there's only one.** "One positive integer is ___ more than twice another, and they sum to 28" becomes $2x + y + x = 28$, i.e. $3x + y = 28$. Don't panic that one equation holds two unknowns: with integer constraints, that's a solvable problem, not a broken one. (Example 2 lives entirely on this line.)

> **2 · Parity check before any work.** $2x + 4y = 41$: the left side is even + even, the right side is odd — **no solutions**, done in five seconds. Always glance at even/odd (or a quick mod) before you start enumerating; the instructor calls this the free win.

> **3 · Push divisibility through the equation.** In $2x + 3y = 45$: since $3 \mid 45$ and $3 \mid 3y$, subtracting gives $3 \mid 2x$, so $3 \mid x$. Now enumerate only $x = 0, 3, 6, \ldots$ with $y = 15, 13, 11, \ldots$ Stronger version: take the whole equation mod a coefficient. $5x + 6y = 49 \pmod 5$: $6y \equiv y$ and $49 \equiv 4$, so $y \equiv 4 \pmod 5$ — try $y = 4$ and $x = 5$ pops out.

4 · One solution, then step by coefficients. $5x + y = 50$: spot $(x, y) = (0, 50)$; each $+1$ in x costs -5 in y: $(0, 50) \to (1, 45) \to (2, 40) \to \cdots$ In general $x = k,\ y = 50 - 5k$. Written from the other end: $x = 10 - k,\ y = 5k$. Same ladder, two starting rungs.

$$(0, 50) \xrightarrow{x+1,\ y-5} (1, 45) \xrightarrow{x+1,\ y-5} (2, 40)$$

in ax+by=c the steps are x→x+b, y→y−a

5 · Count solutions with the bounds. From $y = 50 - 5k \geq 0$: $k \leq 10$, so $k = 0, 1, \ldots, 10$ — 11 solutions. Want only $30 \leq y \leq 40$? Then $30 \leq 50 - 5k \leq 40 \Rightarrow -20 \leq -5k \leq -10 \Rightarrow 2 \leq k \leq 4$ — 3 solutions. **Careful:** dividing by a negative flips the inequality — the board circles that flip in red.

6 · Two equations, three unknowns → eliminate down to one Diophantine line. The socks system $x + y + z = 12$, $x + 3y + 4z = 24$: subtract to kill x, leaving $2y + 3z = 12$; then bounds + parity finish it (Example 3). Getting rid of one variable turns a scary system into tip #4.

7 · Fractions of an unknown total → the total is a multiple of the LCM of the denominators. If $\frac{1}{3}$ of the marbles are blue and $\frac{1}{4}$ are red, the total is a multiple of 12 (Example 1). If $\frac{1}{4}x + \frac{2}{7}x$ must be whole and bounded, x moves in steps of 28 — that plus a bound pins $x = 56$ in Example 4. The denominators tell you the step size.

8 · GCF/LCM mechanics, three ways.

- **Short division:** strip common factors: $2 \mid (12, 18) \to 3 \mid (6, 9) \to (2, 3)$. GCF = the strips: $2 \cdot 3 = 6$; LCM = strips × leftovers: $2 \cdot 3 \cdot 2 \cdot 3 = 36$.
- **Prime powers:** GCF takes the smaller exponent of each prime, LCM the larger. ($180 = 2^2 \cdot 3^2 \cdot 5$, $594 = 2 \cdot 3^3 \cdot 11$ → GCF $2 \cdot 3^2$, LCM $2^2 \cdot 3^3 \cdot 5 \cdot 11$.)
- **The identity:** $\gcd(A, B) \cdot \operatorname{lcm}(A, B) = AB$; and if $A = mx$, $B = nx$ with $\gcd(m, n) = 1$, then $\gcd = x$ and $\operatorname{lcm} = mnx$.

9 · Same remainder, several divisors → LCM plus the remainder. "Leaves remainder 2 when divided by $3, 4, 5, 6$" means $N - 2$ is a multiple of $\operatorname{lcm}(3, 4, 5, 6) = 60$, so $N = 62$ (Practice 3). "Has 15, 20, and 25 as factors" means a multiple of $\operatorname{lcm} = 300$ (Practice 4). Board bonus: know the divisors of 60 cold — $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$ — they unlock every minutes-per-mile and clock problem.

📖 Worked examples

2017 AMC 8 #9 · LCM of denominators

Example 1. All of Marcy's marbles are blue, red, green, or yellow. One third of her marbles are blue, one fourth of them are red, and six of them are green. What is the smallest number of yellow marbles?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

🧰 #7 LCM of denominators #1 Write the equation

Key idea: with x marbles total and y yellow, $\frac{1}{3}x + \frac{1}{4}x + 6 + y = x$, so $y = \frac{5}{12}x - 6$. For $\frac{1}{3}x$ and $\frac{1}{4}x$ to be whole numbers, x must be a multiple of 12; and $y > 0$ needs $\frac{5}{12}x > 6$, i.e. $x > 14.4$. The first multiple of 12 past that is $x = 24$, giving $y = \frac{5}{12} \cdot 24 - 6 = 4$. **Answer: D**

2022 AMC 8 #13 · set up & count

Example 2. How many positive integers can fill the blank in the sentence below? "One positive integer is ___ more than twice another, and the sum of the two numbers is 28."

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

🧰 #1 Write the equation #5 Count with bounds

Key idea: let the smaller number be x and the blank be y: the other number is $2x + y$, and $(2x + y) + x = 28$, i.e. $3x + y = 28$. Each $x = 1, 2, \ldots, 9$ gives exactly one positive $y = 28 - 3x$ ($x = 9 \Rightarrow y = 1$; $x = 10$ would force $y < 0$). That's 9 possible blanks. **Answer: D**

2015 AMC 8 #20 · system → one line

Example 3. Ralph went to the store and bought 12 pairs of socks for a total of $\$24$. Some of the socks he bought cost $\$1$ a pair, some of the socks he bought cost $\$3$ a pair, and some of the socks he bought cost $\$4$ a pair. If he bought at least one pair of each type, how many pairs of $\$1$ socks did Ralph buy?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

#6 Eliminate · #2 Parity · #5 Count with bounds

Key idea: with x, y, z pairs at $\$1, \$3, \$4$: $x + y + z = 12$ and $x + 3y + 4z = 24$. Subtract: $2y + 3z = 12$. Parity: $3z$ must be even, so z is even; $z \geq 1$ forces $z = 2$ ($z = 4$ gives $y = 0$, not allowed). Then $y = 3$ and $x = 12 - 3 - 2 = 7$. **Answer: D**

2019 AMC 8 #23 · bounds + integrality

Example 4. After Euclid High School's last basketball game, it was determined that $\frac{1}{4}$ of the team's points were scored by Alexa and $\frac{2}{7}$ were scored by Brittany. Chelsea scored 15 points. None of the other 7 team members scored more than 2 points. What was the total number of points scored by the other 7 team members?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

#7 LCM of denominators · #5 Count with bounds

Key idea: let the team total be x. The other seven scored $x - \frac{1}{4}x - \frac{2}{7}x - 15 = \frac{13}{28}x - 15$, which must be an integer between 0 and 14. Integrality forces $28 \mid x$ (steps of 28!), and $0 \leq \frac{13}{28}x - 15 \leq 14$ pins $x = 56$. Then the seven scored $\frac{13}{28} \cdot 56 - 15 = 26 - 15 = 11$. **Answer: B**

2016 AMC 8 #20 · LCM constraints

Example 5. The least common multiple of a and b is 12, and the least common multiple of b and c is 15. What is the least possible value of the least common multiple of a and c?

(A) 20 **(B)** 30 **(C)** 60 **(D)** 120 **(E)** 180

🧰 #8 GCF/LCM mechanics

Key idea: $\operatorname{lcm}(a, b) = 2^2 \cdot 3$ and $\operatorname{lcm}(b, c) = 3 \cdot 5$. The factor 2^2 can only come from a (since b would push 2 into $\operatorname{lcm}(b, c)$), and the 5 can only come from c. So $4 \mid a$ and $5 \mid c$. Choose the leanest: $a = 4$, $c = 5$ (with $b = 3$ making both LCMs work): $\operatorname{lcm}(a, c) = 20$.

Answer: A

✏️ Practice

Try each one before opening the solution. Figures are from the official exams.

2016 AMC 8 #11

1. Determine how many two-digit numbers satisfy the following property: when the number is added to the number obtained by reversing its digits, the sum is 132.

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 12

2013 AMC 8 #10

2. What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?

(A) 110 **(B)** 165 **(C)** 330 **(D)** 625 **(E)** 660

2012 AMC 8 #15

3. The smallest number greater than 2 that leaves a remainder of 2 when divided by 3, 4, 5, or 6 lies between what numbers?

(A) 40 and 50 **(B)** 51 and 55 **(C)** 56 and 60 **(D)** 61 and 65 **(E)** 66 and 99

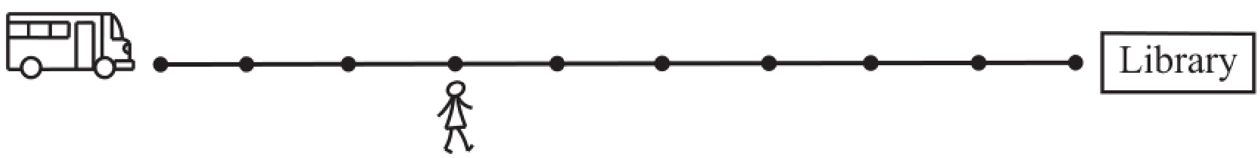
2003 AMC 8 #19

4. How many integers between 1000 and 2000 have all three of the numbers 15, 20, and 25 as factors?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2022 AMC 8 #22

5. A bus takes 2 minutes to drive from one stop to the next, and waits 1 minute at each stop to let passengers board. Zia takes 5 minutes to walk from one bus stop to the next. As Zia reaches a bus stop, if the bus is at the previous stop or has already left the previous stop, then she will wait for the bus. Otherwise she will start walking toward the next stop. Suppose the bus and Zia start at the same time toward the library, with the bus 3 stops behind. After how many minutes will Zia board the bus?



(A) 17 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 23

🆕 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2025 AMC 8 #6

6. Sekou writes down the numbers $15, 16, 17, 18, 19$. After he erases one of his numbers, the sum of the remaining four numbers is a multiple of 4. Which number did he erase?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

2026 AMC 8 #18

7. In how many ways can 60 be written as the sum of two or more consecutive odd positive integers that are arranged in increasing order?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2024 AMC 8 #15

8. Let the letters F, L, Y, B, U, G represent distinct digits. Suppose $\underline{F}\,\underline{L}\,\underline{Y}\,\underline{F}\,\underline{L}\,\underline{Y}$ is the GREATEST number that satisfies the equation

$$8 \cdot \underline{F}\,\underline{L}\,\underline{Y}\,\underline{F}\,\underline{L}\,\underline{Y} = \underline{B}\,\underline{U}\,\underline{G}\,\underline{B}\,\underline{U}\,\underline{G}.$$

What is the value of $\underline{F}\,\underline{L}\,\underline{Y} + \underline{B}\,\underline{U}\,\underline{G}$?

(A) 1089 **(B)** 1098 **(C)** 1107 **(D)** 1116 **(E)** 1125

⭐ Challenge (AMC 10 level)

2017 AMC 10A #16 · LCM

C1. There are 10 horses, named Horse 1, Horse $2, \ldots,$ Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point. The horses run in the same direction at their constant speeds. The least time $S > 0$, in minutes, at which *all* 10 horses are again simultaneously at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that *at least* 5 of the horses are again at the starting point. What is the sum of the digits of T?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Sources: prep-course Lecture 16 (video + board notes + 11 worked examples, translated to English) and official AMC 8 / AMC 10 problems from the local bank (`kb/content/banks/`). Answers verified against the bank's official answer keys where available.

🗝️ Answer key

Example	1	2	3	4	5			
	D	D	D	B	A			
Practice	1	2	3	4	5	6	7	8
	B	C	D	C	A	C	B	C
Challenge	C1							
	B							

📝 Solutions to practice problems

Practice 1.

🧰 #1 Write the equation #5 Count with bounds

Write the number as $10a + b$; reversed it is $10b + a$. Then $(10a + b) + (10b + a) = 11(a + b) = 132$, so $a + b = 12$. Digits run 1–9 here (with $a + b = 12$ neither digit can be 0), so the pairs are $(3, 9), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4), (9, 3)$ — 7 numbers. **Answer: B**

Practice 2.

🧰 #8 GCF/LCM mechanics

$180 = 2^2 \cdot 3^2 \cdot 5$ and $594 = 2 \cdot 3^3 \cdot 11$. LCM (larger exponents) $= 2^2 \cdot 3^3 \cdot 5 \cdot 11$; GCF (smaller exponents) $= 2 \cdot 3^2$. Ratio $= \dfrac{2^2 \cdot 3^3 \cdot 5 \cdot 11}{2 \cdot 3^2} = 2 \cdot 3 \cdot 5 \cdot 11 = 330$. **Answer: C**

Practice 3.

🧰 #9 LCM + remainder

$N - 2$ is divisible by $3, 4, 5, 6$, so $N - 2$ is a multiple of $\operatorname{lcm}(3, 4, 5, 6) = 60$. Smallest: $N = 62$, which lies between 61 and 65. **Answer: D**

Practice 4.

🧰 #9 LCM + remainder | #8 GCF/LCM mechanics

Such a number is a multiple of $\operatorname{lcm}(15, 20, 25)$. With $15 = 3 \cdot 5$, $20 = 2^2 \cdot 5$, $25 = 5^2$: $\operatorname{lcm} = 2^2 \cdot 3 \cdot 5^2 = 300$. Between 1000 and 2000: $1200, 1500, 1800$ — three of them.
Answer: C

Practice 5.

🧰 #9 LCM + remainder

The bus advances one stop every 3 minutes (drive 2, wait 1); Zia advances one stop every 5 minutes. So check the clock at $\operatorname{lcm}(3, 5) = 15$ minutes: the bus has made 5 hops and Zia 3 — starting 3 stops back, the bus is now exactly one stop behind her, waiting to depart. By the rule Zia stays put, and the bus needs 2 more minutes of driving to reach her: $15 + 2 = 17$.
Answer: A

Practice 6.

🧰 #2 Parity / mod check

All five sum to 85, and $85 = 84 + 1 \equiv 1 \pmod 4$. For the rest to be a multiple of 4, the erased number must also be $\equiv 1 \pmod 4$ — that's 17. **Answer: C**

Practice 7.

🧰 #1 Write the equation | #3 Divisibility | #5 Count with bounds

Let $k \geq 2$ odds start at a: the sum is $ka + 2(0 + 1 + \cdots + (k-1)) = k(a + k - 1) = 60$. So $k \mid 60$ and $a = \frac{60}{k} - k + 1$ must be a positive *odd* integer. Run the divisors: $k = 2 \Rightarrow a = 29$ ✓ $(29 + 31)$; $k = 3 \Rightarrow 18$ ✗; $k = 4 \Rightarrow 12$ ✗; $k = 5 \Rightarrow 8$ ✗; $k = 6 \Rightarrow 5$ ✓ ($5 + 7 + 9 + 11 + 13 + 15$); $k = 10 \Rightarrow a < 0$. Two ways. **Answer: B**

Practice 8.

🧰 #8 Factor structure #5 Count with bounds

The repeat pattern is a factorization in disguise: $\underline{FLYFLY} = 1001 \cdot \underline{FLY}$ and $\underline{BUGBUG} = 1001 \cdot \underline{BUG}$. Cancel 1001: the equation is just $8 \cdot \underline{FLY} = \underline{BUG}$. For $\underline{BUG}$ to stay three digits, $\underline{FLY} \le 124$. Test downward with distinct digits: $124 \to 992$ (repeated 9) ✗; $123 \to 984$ — digits $1, 2, 3, 9, 8, 4$ all distinct ✓. $123 + 984 = 1107$.

Answer: C

Practice C1.

🧰 #9 LCM + remainder #8 GCF/LCM mechanics

Horse k is at the start exactly at the multiples of k, so T works iff at least 5 of the numbers $1, \ldots, 10$ divide T — i.e. T has 5 divisors that are ≤ 10. Hunt for the smallest such T: 12 is divisible by $1, 2, 3, 4, 6$ — five horses — and no smaller number has five divisors. $T = 12$, digit sum $1 + 2 = 3$. **Answer: B**