

Solving Equations · Lecture 07 · alg-equations

 **Course materials** — **Lecture 07**: lecture video · handout PDF (Lecture 7) · board notes (07.jpg) · homework answer key (Chapter 7) — all in the `amc8/video` folder. ·  [Printable PDF](#) (answers & solutions at the back)

Key facts

Almost every AMC 8 word problem is an equation in disguise. The whole game is **translation**: name the unknown (x), turn each sentence of the story into one equation, solve, and answer *the question that was asked* (not the value of x !). Three translations carry the chapter: ratio $3 : 4 \rightarrow$ parts $3k$ and $4k$; percent change $\rightarrow$ multiply by $(1 \pm p\%)$; and average speed = total distance $\div$ total time.

Instructor's toolbox (from the lecture video & board notes)

1 · One sentence $\rightarrow$ one equation. Count the conditions: one condition, one unknown; two conditions, set two unknowns and build a system. The instructor's two exits from a system: **elimination** — add or subtract the equations ($A + B = 20$, $A - B = 2 \rightarrow$ add: $2A = 22$) — and **substitution** — solve one equation for a variable and plug it into the other ($A = B + 2$ into $A + B = 20$). Example 3 (gold coins) is pure substitution; Practice 14 turns a story straight into a quadratic.

2 · Ratios are parts. Heights in ratio $3 : 4$? Write them as $3k$ and $4k$ — then "one tree is 16 feet taller" becomes $4k - 3k = 16$, so $k = 16$ (Practice 3). And keep *part-to-part* separate from *part-to-whole*: if $A : B = 30 : 20$, then A is $\frac{30}{30 + 20} = \frac{3}{5}$ of the total — the board opens with exactly this part-vs-whole warning.

3 · Chain two ratios with the LCM. To merge $8^{\text{th}} : 6^{\text{th}} = 5 : 3$ and $8^{\text{th}} : 7^{\text{th}} = 8 : 5$ into one continued ratio, rescale the *shared* member to $\text{lcm}(5, 8) = 40$: multiply the first ratio by 8 and the second by 5, giving $6^{\text{th}} : 7^{\text{th}} : 8^{\text{th}} = 24 : 25 : 40$. Each part is at least one whole person, so the minimum total is $24 + 25 + 40 = 89$ (Example 2). The board's motto: *find the common quantity and unify it*.

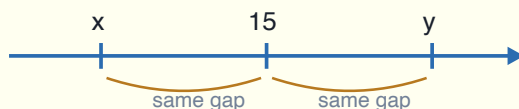
4 · Find what doesn't change. When a ratio shifts, anchor the parts to the **invariant**. Ages: the difference never changes, so give the difference the same number of parts before and after. Two people whose ages sum to a constant: rescale both ratios so the *sum* of parts matches (the board rescales $3 : 4$ by 8 and $5 : 3$ by 7 so both totals are 56). In Practice 4 the green and orange socks are the invariant — only purple grows.

5 • Percent changes multiply — and mind the base. Start at a : up $x\%$, up $y\%$, down $z\%$ lands on $a(1 + x\%)(1 + y\%)(1 - z\%)$ — never add the percents. Up then down by the same y : $(1 + y)(1 - y) = 1 - y^2$, always a net *loss* (Example 4). And every percent is measured against its *own* base, the value right before the change: $20 \rightarrow 50$ is a $\frac{30}{20} = 150\%$ increase, but $50 \rightarrow 20$ is only a $\frac{30}{50} = 60\%$ decrease — the instructor draws both arrows side by side.

6 • Assume a convenient total. When everything is percents or fractions and no count is given, invent one: 100 marbles (Example 1), 200 homework problems so each half is 100 (Practice 7), 100 campers (Challenge C1). Percents become plain counts and the arithmetic does itself.

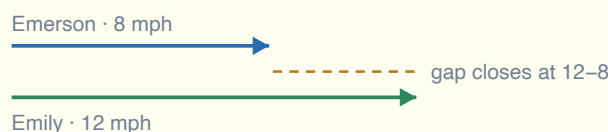
7 • Percent sentence $\rightarrow$ fraction equation $\rightarrow$ cross-multiply. "60% of the drawer is purple" becomes $\frac{18 + x}{36 + x} = \frac{3}{5}$, and cross-multiplying gives $5(18 + x) = 3(36 + x)$, so $x = 9$ (Practice 4). Put the unknown in the fraction first; clear denominators immediately.

8 • Equally spaced = average. If x , 15, y are equally spaced on a number line, the middle one is the mean: $x + y = 2 \cdot 15 = 30$ (Practice 2). Any arithmetic sequence works this way — the middle term is the average of the two ends.



9 • Motion toolkit.

- Average speed = total distance $\div$ total time** — never the average of the speeds. Qiang: $\frac{15 + x}{0.5 + \frac{x}{55}} = 50$ (Practice 11).
- Chasing:** freeze the slower one — the gap closes at $v_1 - v_2$. Emily gains on Emerson at $12 - 8 = 4$ mph, so a 1-mile window takes $\frac{1}{4}$ h (Example 5). **Meeting** head-on: distances add, so use $v_1 + v_2$.
- On a **distance–time graph**, speed is the *slope* — steeper line, faster runner (the board opens the lecture with the four slope cases: positive, zero, negative, undefined).



10 • Comparing fractions — four honest ways, one forbidden one. For $\frac{5}{19}$, $\frac{7}{21}$, $\frac{9}{23}$ the instructor shows: **reciprocals** — flip everything: $\frac{19}{5} = 3.8 > \frac{21}{7} = 3 > \frac{23}{9} \approx 2.6$, so the original order reverses to $\frac{5}{19} < \frac{7}{21} < \frac{9}{23}$; **benchmark** — compare each to $\frac{1}{3}$ ($\frac{5}{19} < \frac{5}{15} = \frac{1}{3}$, $\frac{9}{23} > \frac{9}{27} = \frac{1}{3}$); **decimal estimate** — 0.26, 0.33, 0.39. Forbidden: adding tops and bottoms — $\frac{1+2}{3+5}$ is *not* how fractions add (the board crosses it out in red).

Worked examples

2019 AMC 8 #8 · successive percents

Example 1. Gilda has a bag of marbles. She gives 20% of them to her friend Pedro. Then Gilda gives 10% of what is left to another friend, Ebony. Finally, Gilda gives 25% of what is now left in the bag to her brother Jimmy. What percentage of her original bag of marbles does Gilda have left for herself?

- (A) 20 (B) $33\frac{1}{3}$ (C) 38 (D) 45 (E) 54



#6 Assume a total

#5 Percents multiply

Key idea: start with 100 marbles and just follow the story. After Pedro: $100(1 - 20\%) = 80$. After Ebony: $80(1 - 10\%) = 72$. After Jimmy: $72(1 - 25\%) = 54$. That's 54 of the original 100 — each percent acted on the *current* amount, not the original. **Answer: E**

2013 AMC 8 #16 · continued ratio

Example 2. A number of students from Fibonacci Middle School are taking part in a community service project. The ratio of 8th-graders to 6th-graders is 5 : 3, and the ratio of 8th-graders to 7th-graders is 8 : 5. What is the smallest number of students that could be participating in the project?

- (A) 16 (B) 40 (C) 55 (D) 79 (E) 89



#3 Chain ratios (LCM)

#2 Ratios are parts

Key idea: the 8th grade appears in both ratios with different numbers (5 and 8) — unify it at $\text{lcm}(5, 8) = 40$. $8^{\text{th}} : 6^{\text{th}} = 5 : 3 = 40 : 24$ and $8^{\text{th}} : 7^{\text{th}} = 8 : 5 = 40 : 25$, so $6^{\text{th}} : 7^{\text{th}} : 8^{\text{th}} = 24 : 25 : 40$. Each part is at least one student, so the minimum is $24 + 25 + 40 = 89$.

Answer: E

Example 3. Starting with some gold coins and some empty treasure

chests, I tried to put 9 gold coins in each treasure chest, but that left 2 treasure chests empty. So instead I put 6 gold coins in each treasure chest, but then I had 3 gold coins left over. How many gold coins did I have?

- (A) 9 (B) 27 (C) 45 (D) 63 (E) 81



#1 One sentence, one equation

Key idea: two sentences, two equations. Let x = coins, y = chests. "Nine per chest, two chests empty": $x = 9(y - 2)$. "Six per chest, three left over": $x = 6y + 3$. Substitute: $9(y - 2) = 6y + 3 \Rightarrow 9y - 18 = 6y + 3 \Rightarrow 3y = 21 \Rightarrow y = 7$, so $x = 6 \cdot 7 + 3 = 45$. **Answer: C**

Example 4. A store increased the original price of a shirt by a certain

percent and then decreased the new price by the same percent. Given that the resulting price was 84% of the original price, by what percent was the price increased and decreased?

- (A) 16 (B) 20 (C) 28 (D) 36 (E) 40



#5 Percents multiply

#1 One sentence, one equation

Key idea: up p then down p multiplies the price by $(1 + p)(1 - p) = 1 - p^2$. So $1 - p^2 = 0.84 \Rightarrow p^2 = 0.16 \Rightarrow p = 0.4 = 40\%$. (Sanity check from the board: $100 \rightarrow 140 \rightarrow 84$. ✓ And note the answer is *not* 16% — that trap is choice A.) **Answer: E**

Example 5. As Emily is riding her bicycle on a long straight road, she spots

Emerson skating in the same direction $\frac{1}{2}$ mile in front of her. After she passes him, she can see him in her rear-view mirror until he is $\frac{1}{2}$ mile behind her. Emily rides at a constant rate of 12 miles per hour, and Emerson skates at a constant rate of 8 miles per hour. For how many minutes can Emily see Emerson?

- (A) 6 (B) 8 (C) 12 (D) 15 (E) 16



#9 Motion toolkit

Key idea: freeze Emerson. Relative to him, Emily moves at $12 - 8 = 4$ mph. She must travel from $\frac{1}{2}$ mile behind him to $\frac{1}{2}$ mile ahead — a relative distance of 1 mile. Time = $\frac{1}{4}$ h = 15 minutes. **Answer:**

D

Practice

Try each one before opening the solution. The set walks from ratios through percents to motion — same order as the lecture.

2020 AMC 8 #1

1. Luka is making lemonade to sell at a school fundraiser. His recipe requires 4 times as much water as sugar and twice as much sugar as lemon juice. He uses 3 cups of lemon juice. How many cups of water does he need?

- (A) 6 (B) 8 (C) 12 (D) 18 (E) 24

2022 AMC 8 #6

2. Three positive integers are equally spaced on a number line. The middle number is 15, and the largest number is 4 times the smallest number. What is the smallest of these three numbers?

- (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

2010 AMC 8 #11

3. The top of one tree is 16 feet higher than the top of another tree. The heights of the two trees are in the ratio 3 : 4. In feet, how tall is the taller tree?

- (A) 48 (B) 64 (C) 80 (D) 96 (E) 112

2020 AMC 8 #13

4. Jamal has a drawer containing 6 green socks, 18 purple socks, and 12 orange socks. After adding more purple socks, Jamal noticed that there is now a 60% chance that a sock randomly selected from the drawer is purple. How many purple socks did Jamal add?

- (A) 6 (B) 9 (C) 12 (D) 18 (E) 24

2020 AMC 8 #15

5. Suppose 15% of x equals 20% of y . What percentage of x is y ?

- (A) 5 (B) 35 (C) 75 (D) $133\frac{1}{3}$ (E) 300

2012 AMC 8 #20

6. What is the correct ordering of the three numbers $\frac{5}{19}$, $\frac{7}{21}$, and $\frac{9}{23}$, in increasing order?

- (A) $\frac{9}{23} < \frac{7}{21} < \frac{5}{19}$ (B) $\frac{5}{19} < \frac{7}{21} < \frac{9}{23}$ (C) $\frac{9}{23} < \frac{5}{19} < \frac{7}{21}$ (D) $\frac{5}{19} < \frac{9}{23} < \frac{7}{21}$ (E) $\frac{7}{21} < \frac{5}{19} < \frac{9}{23}$

2017 AMC 8 #14

7. Chloe and Zoe are both students in Ms. Demeanor's math class. Last night, they each solved half of the problems in their homework assignment alone and then solved the other half together. Chloe had correct answers to only 80% of the problems she solved alone, but overall 88% of her answers were correct. Zoe had correct answers to 90% of the problems she solved alone. What was Zoe's overall percentage of correct answers?

- (A) 89 (B) 92 (C) 93 (D) 96 (E) 98

2022 AMC 8 #16

8. Four numbers are written in a row. The average of the first two is 21, the average of the middle two is 26, and the average of the last two is 30. What is the average of the first and last of the numbers?

- (A) 24 (B) 25 (C) 26 (D) 27 (E) 28

2015 AMC 8 #16

9. In a middle-school mentoring program, a number of the sixth graders are paired with a ninth-grade student as a buddy. No ninth grader is assigned more than one sixth-grade buddy. If $\frac{1}{3}$ of all the ninth graders are paired with $\frac{2}{5}$ of all the sixth graders, what fraction of the total number of sixth and ninth graders have a buddy?

- (A) $\frac{2}{15}$ (B) $\frac{4}{11}$ (C) $\frac{11}{30}$ (D) $\frac{3}{8}$ (E) $\frac{11}{15}$

2018 AMC 8 #6

10. On a trip to the beach, Anh traveled 50 miles on the highway and 10 miles on a coastal access road. He drove three times as fast on the highway as on the coastal road. If Anh spent 30 minutes driving on the coastal road, how many minutes did his entire trip take?

- (A) 50 (B) 70 (C) 80 (D) 90 (E) 100

2019 AMC 8 #16

11. Qiang drives 15 miles at an average speed of 30 miles per hour. How many additional miles will he have to drive at 55 miles per hour to average 50 miles per hour for the entire trip?

- (A) 45 (B) 62 (C) 90 (D) 110 (E) 135

2018 AMC 8 #12

12. The clock in Sri's car, which is not accurate, gains time at a constant rate. One day as he begins shopping, he notes that his car clock and his watch (which is accurate) both say 12:00 noon. When he is done shopping, his watch says 12:30 and his car clock says 12:35. Later that day, Sri loses his watch. He looks at his car clock and it says 7:00. What is the actual time?

- (A) 5:50 (B) 6:00 (C) 6:30 (D) 6:55 (E) 8:10

2019 AMC 8 #20

13. How many different real numbers x satisfy the equation

$$(x^2 - 5)^2 = 16?$$

- (A) 0 (B) 1 (C) 2 (D) 4 (E) 8

2009 AMC 8 #23

14. On the last day of school, Mrs. Awesome gave jelly beans to her class. She gave each boy as many jelly beans as there were boys in the class. She gave each girl as many jelly beans as there were girls in the class. She brought 400 jelly beans, and when she finished, she had 6 jelly beans left. There were two more boys than girls in her class. How many students were in her class?

- (A) 26 (B) 28 (C) 30 (D) 32 (E) 34

 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2026 AMC 8 #4

15. Brynn's savings decreased by 20% in July, then increased by 50% in August. Brynn's savings are now what percent of the original amount?

- (A) 80 (B) 90 (C) 100 (D) 110 (E) 120

2026 AMC 8 #5

16. Casey went on a road trip that covered 100 miles, stopping for a lunch break along the way. The trip took 3 hours in total and her average speed while driving was 40 miles per hour (mph). In minutes, how long was the lunch break?

- (A) 15 (B) 30 (C) 40 (D) 45 (E) 60

2024 AMC 8 #21

17. A group of frogs (called an army) is living in a tree. A frog turns green when in the shade and turns yellow when in the sun. Initially, the ratio of green to yellow frogs was 3 : 1. Then 3 green frogs moved to the sunny side and 5 yellow frogs moved to the shady side. Now the ratio is 4 : 1. What is the difference between the number of green frogs and the number of yellow frogs now?

- (A) 10 (B) 12 (C) 16 (D) 20 (E) 24

★ Challenge (AMC 10 level)

2009 AMC 10A #18 · percents + Venn

C1. At Jefferson Summer Camp, 60% of the children play soccer, 30% of the children swim, and 40% of the soccer players swim. To the nearest whole percent, what percent of the non-swimmers play soccer?

- (A) 30% (B) 40% (C) 49% (D) 51% (E) 70%

2002 AMC 10A #12 · system + motion

C2. Mr. Earl E. Bird leaves home every day at 8:00 AM to go to work. If he drives at an average speed of 40 miles per hour, he will be late by 3 minutes. If he drives at an average speed of 60 miles per hour, he will be early by 3 minutes. How many miles per hour does Mr. Bird need to drive to get to work exactly on time?

- (A) 45 (B) 48 (C) 50 (D) 55 (E) 58

Sources: prep-course Lecture 07 (video + board notes + 36 worked examples, translated to English and curated to the most instructive) and official AMC 8 problems from the local bank (kb/content/banks/). Answers cross-checked between the course solutions and the bank's official answer keys.

Answer key

Example	1	2	3	4	5												
	E	E	C	E	D												
Practice	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	E	C	B	B	C	B	C	B	B	C	D	B	D	B	E	B	E
Challenge	C1	C2															
	D	B															

Solutions to practice problems

Practice 1.



#2 Ratios are parts

#3 Chain ratios

Chain the two comparisons: water : sugar : lemon juice = 8 : 2 : 1. Lemon juice is 3 cups, so water = $3 \cdot 8 = 24$. **Answer: E**

Practice 2.



#8 Equally spaced = average

#1 One sentence, one equation

Equally spaced means the middle is the mean: smallest + largest = $2 \cdot 15 = 30$. With smallest x and largest $4x$: $x + 4x = 30 \Rightarrow x = 6$. **Answer: C**

Practice 3.



#2 Ratios are parts

Heights $3k$ and $4k$: the difference is one part, $4k - 3k = k = 16$. Taller tree = $4k = 64$ feet.

Answer: B

Practice 4.



#7 Cross-multiply

#4 Find the invariant

The $6 + 12 = 18$ non-purple socks never change. With x new purple socks, $\frac{18 + x}{36 + x} = \frac{3}{5}$. Cross-multiply: $5(18 + x) = 3(36 + x) \Rightarrow 90 + 5x = 108 + 3x \Rightarrow x = 9$. **Answer: B**

Practice 5.



#7 Cross-multiply

$0.15x = 0.2y \Rightarrow y = \frac{0.15}{0.2}x = 0.75x$ — so y is 75% of x . (The board's shortcut: divide both sides by 20%, leaving $x \cdot \frac{15}{20} = y$.) **Answer: C**

Practice 6.



#10 Comparing fractions

Flip to reciprocals: $\frac{19}{5} = 3.8$, $\frac{21}{7} = 3$, $\frac{23}{9} \approx 2.6$. Bigger reciprocal = smaller fraction, so $\frac{5}{19} < \frac{7}{21} < \frac{9}{23}$. (Benchmark check: $\frac{7}{21} = \frac{1}{3}$ exactly; $\frac{5}{19} < \frac{1}{3} < \frac{9}{23}$.) **Answer: B**

Practice 7.



#6 Assume a total

Say the assignment has 200 problems — 100 alone, 100 together. Chloe: 80 right alone, and $\frac{80 + x}{200} = 88\%$ gives $x = 96$ correct in the joint half (shared with Zoe). Zoe: $\frac{90 + 96}{200} = \frac{186}{200} = 93\%$. **Answer: C**

Practice 8.



#1 One sentence, one equation

$a + b = 42$, $b + c = 52$, $c + d = 60$. Add the first and last, subtract the middle: $a + d = (a + b) + (c + d) - (b + c) = 42 + 60 - 52 = 50$, so the average is 25. No need to find any individual number! **Answer: B**

Practice 9.



#2 Ratios are parts

#7 Cross-multiply

Pairs are one-to-one, so $\frac{1}{3}n = \frac{2}{5}s$, giving $\frac{n}{s} = \frac{6}{5}$: say $n = 6k$ ninth graders and $s = 5k$ sixth graders.

Buddied students: $\frac{1}{3} \cdot 6k + \frac{2}{5} \cdot 5k = 2k + 2k = 4k$ out of $11k$ total = $\frac{4}{11}$. **Answer: B**

Practice 10.



#9 Motion toolkit

Coastal: 10 miles in 30 min = $\frac{1}{3}$ mile per minute. Highway: $3\times$ faster = 1 mile per minute, so 50 miles takes 50 minutes. Total $30 + 50 = 80$ minutes. **Answer: C**

Practice 11.



#9 Motion toolkit

#7 Cross-multiply

Average speed is total distance over total time — set up exactly that: $\frac{15 + x}{0.5 + \frac{x}{55}} = 50$ (the first leg takes

$\frac{15}{30} = 0.5$ h). Then $15 + x = 25 + \frac{10x}{11} \Rightarrow \frac{x}{11} = 10 \Rightarrow x = 110$. **Answer: D**

Practice 12.



#2 Ratios are parts

#9 Motion toolkit

Treat the two clocks as runners at constant speeds: real : car = $30 : 35 = 6 : 7$. When the car clock has "run" 7 hours (noon $\rightarrow$ 7:00), real time has run 6 hours — it's 6:00 PM. **Answer: B**

Practice 13.



#1 One sentence, one equation

A square equal to 16 can be $+4$ or -4 : $x^2 - 5 = \pm 4$, so $x^2 = 9$ or $x^2 = 1$, giving $x = \pm 3, \pm 1$ — four real solutions. (Forgetting the negative branch is the whole trap.) **Answer: D**

Practice 14.



#1 One sentence, one equation

Girls x , boys $x + 2$; the handout used $400 - 6 = 394$ jelly beans: $x^2 + (x + 2)^2 = 394 \Rightarrow 2x^2 + 4x - 390 = 0 \Rightarrow x^2 + 2x - 195 = 0 \Rightarrow (x - 13)(x + 15) = 0$. So $x = 13$: 13 girls + 15 boys = 28 students. **Answer: B**

Practice 15.



#5 Percents multiply

Multiply the factors: $(1 - 20\%)(1 + 50\%) = 0.8 \times 1.5 = 1.2$ — that's 120% of the original. (Adding $-20\% + 50\% = +30\%$ would be wrong: each percent has its own base.) **Answer: E**

Practice 16.



#9 Motion toolkit

Driving time = $\frac{100}{40} = 2.5$ h. The trip took 3 h, so lunch was 0.5 h = 30 minutes. **Answer: B**

Practice 17.



#2 Ratios are parts

#1 One sentence, one equation

Start with $3x$ green, x yellow. After the swap: green $3x - 3 + 5 = 3x + 2$, yellow $x + 3 - 5 = x - 2$. New ratio: $3x + 2 = 4(x - 2) \Rightarrow x = 10$. Now green = 32, yellow = 8, difference = 24. **Answer: E**

Practice C1.



#6 Assume a total

Take 100 campers: 60 play soccer, 30 swim, and $60 \cdot 40\% = 24$ do both. Non-swimmers: $100 - 30 = 70$; of these, soccer players are $60 - 24 = 36$. $\frac{36}{70} \approx 51.4\% \rightarrow 51\%$. **Answer: D**

Practice C2.



#1 One sentence, one equation

#9 Motion toolkit

Let s = distance in miles, t = minutes available. Convert: $40 \text{ mph} = \frac{2}{3} \text{ mi/min}$ and $60 \text{ mph} = 1 \text{ mi/min}$.
The two sentences become $s = \frac{2}{3}(t + 3)$ and $s = 1 \cdot (t - 3)$. Substitute: $\frac{2}{3}(t + 3) = t - 3 \Rightarrow$
 $2t + 6 = 3t - 9 \Rightarrow t = 15, s = 12$. On-time speed: $\frac{12}{15} \text{ mi/min} = 0.8 \cdot 60 = 48 \text{ mph}$. **Answer: B**