Inequalities · Lecture 11 · `alg-inequalities`

📺 **Course materials — Lecture 11 (93 min):** lecture video · handout PDF (Lecture 11) · board notes (11.jpg) · homework answer key (Chapter 11) — all in the `amc8/video` folder. · 🖨️ **Printable PDF** (answers & solutions at the back)

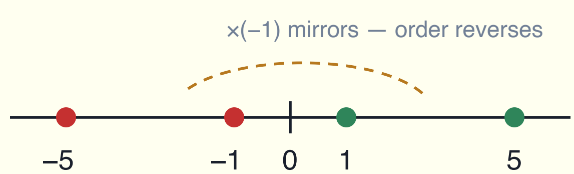
🔑 Key facts

An inequality behaves like an equation with **one exception**: you may add or subtract anything on both sides, and multiply or divide both sides by any **positive** number — but multiplying or dividing by a **negative** number **flips the sign** ($5 > 1$ becomes $-5 < -1$), and multiplying by 0 is never allowed. On the AMC 8 an inequality is usually the *last* step: translate the story into a range like $77.5 < x < 82$, then **count the integers inside** — watching whether the endpoints are included ($1 \le x \le 5$) or not ($1 < x < 5$).

🧰 Instructor's toolbox (from the lecture video & board notes)

1 · Balance moves — and multiply EVERY term. Equations and inequalities allow the same moves: same add, same subtract, same multiply, same divide (never by 0). The instructor's clearing-fractions drill: $\frac{2x}{3} + 5 = 50$, multiply the *whole line* by 3: $2x + 15 = 150$. Trap version: $\frac{2x}{3} \cdot 5 + 6 = 50$ — times 3 gives $10x + 18 = 150$, because *every* term gets the $\times 3$, not just the fraction. Treat a bracketed block as one unit.

2 · Negative number? The sign flips. Adding or subtracting never changes direction ($6 > 5$ stays $5 > 4$ after both drop by 1). Multiplying or dividing by a *positive* keeps direction ($5 > 1$ → $\times 2$ → $10 > 2$). Multiplying or dividing by a *negative* reverses it: $5 > 1$ → $\times(-1)$ → $-5 < -1$. Never multiply by 0 (it erases the inequality) and never divide by 0. Note $\div 3$ is the same move as $\times \frac{1}{3}$.



3 · Endpoints in or out? Read twice, then count. $(1,5)$ means $1<x<5$ — ends **excluded**; $[1,5]$ means $1\le x\le 5$ — ends **included**. When the answer is "how many integers", solve to a range first, then count carefully: $77.5<x<82$ holds the integers $78, 79, 80, 81$ — four of them, and neither endpoint question can be shrugged off.

4 · Averages: think "distance from the average". The instructor's picture for Example 1: put all five scores at the average 82. If the last test rises by $4k$, the four equal tests must drop by k each to keep the balance — so the last score can only be $82+4k$: that is $86, 90, 94, 98$ (and $k=0$ is banned because the last score must be *higher*). No equations needed.

5 · "Which is impossible?" — attack with extreme numbers. To show an option is *possible*, one concrete example is enough — so try wild values: something tiny like $a=\frac{1}{1000}$, something huge like $c=100$. To show an option is *impossible*, find the one-line reason (in Practice 2: c is already bigger than b, so adding positive a keeps $a+c>b$ forever). Check the easy-to-kill options with examples first; what survives is the answer.

6 · Triangle inequality + integers = a counting problem. In an isosceles triangle with legs a and base b, the only inequality that matters is $2a>b$ (the two legs must reach around the base). Combine with the perimeter equation and let integrality do the rest — in Practice 3, $2a+b=23$ forces b odd and $b<11.5$, so $b\in\{1,3,5,7,9,11\}$: six triangles.

7 · "x is the largest" is an inequality in disguise. Name the missing cells with letters, write the row/column equations, then translate the size condition. In Example 2: the equations give $A=14-x$, and "x is larger than A" becomes $x>14-x$, i.e. $x>7$ — so the smallest *integer* is $x=8$. Always finish by checking the winner really beats all the others.

8 · Winning both halves ≠ winning overall — weights matter. The board's shocker: A shoots better than B on three-pointers ($\frac{13}{20}>\frac{3}{5}$) *and* on two-pointers ($\frac{4}{5}>\frac{15}{20}$), yet B's *overall* rate wins, $\frac{18}{25}>\frac{17}{25}$ — because the two categories carry different numbers of attempts. So never average percentages — go back to raw totals (Example 3: $x+y=25$) and keep the per-half comparisons as inequalities.

9 · Bounding $B=(A+B)-A$: pair opposite extremes. Board drill: $10\le A\le 30$ and $20\le A+B\le 50$. Then B is **largest** when $A+B$ is largest and A smallest ($50-10=40$), and **smallest** when $A+B$ is smallest and A largest ($20-30=-10$; if B can't be negative, that floor becomes 0). The classic error is subtracting same-side ends to get $10\le B\le 20$ — the board lists it as the wrong option.

10 · Let divisibility shrink the search. With integer unknowns, look at remainders before brute force. In Example 4, $3N + 4M = 76$: both $4M$ and 76 are multiples of 4, so $3N$ — hence N — must be a multiple of 4. Together with $N > 2M$ and $M > 4$ only a couple of candidates survive ($N = 12, M = 10$ ✗ fails $N > 2M$; $N = 16, M = 7$ ✓).

📖 Worked examples

2018 AMC 8 #13 · average + range

Example 1. Laila took five math tests, each worth a maximum of 100 points. Laila's score on each test was an integer between 0 and 100, inclusive. Laila received the same score on the first four tests, and she received a higher score on the last test. Her average score on the five tests was 82. How many values are possible for Laila's score on the last test?

(A) 4 **(B)** 5 **(C)** 9 **(D)** 10 **(E)** 18

🧰 #4 Distance from average #3 Count the range

Key idea: four tests at x, last test $y > x$, and $4x + y = 5 \cdot 82 = 410$. From $y = 410 - 4x \leq 100$: $x \geq 77.5$; from $y > x$: $410 - 4x > x$, so $x < 82$. The integers are $x = 78, 79, 80, 81$, each giving one y $(98, 94, 90, 86)$ — four values. **Answer: A**

Average view: raise the last test $4k$ above 82 and the four equal tests drop k each: $y = 82 + 4k$, $k = 1..4$.

2022 AMC 8 #20 · largest missing number

Example 2. The grid below is to be filled with integers in such a way that the sum of the numbers in each row and the sum of the numbers in each column are the same. Four numbers are missing. The number x in the lower left corner is larger than the other three missing numbers. What is the smallest possible value of x?

-2	9	5
		-1
x		8

(A) -1 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 9

#7 Largest → inequality

Key idea: the full column $5, -1, 8$ fixes every sum at 12. Call the other missing numbers A (directly above x), B (center), C (right of x): column 1 is $-2 + A + x = 12$ so $x + A = 14$; row 3 gives $x + C = 4$; row 2 gives $A + B = 13$. "x is largest" against A: $x > 14 - x \Rightarrow x > 7$, so try $x = 8$: then $A = 6$, $B = 7$, $C = -4$ — all smaller than 8 ✓. **Answer: D**

2022 AMC 8 #21 · weighted percentages

Example 3. Steph scored 15 baskets out of 20 attempts in the first half of a game, and 10 baskets out of 10 attempts in the second half. Candace took 12 attempts in the first half and 18 attempts in the second. In each half, Steph scored a higher percentage of baskets than Candace. Surprisingly they ended with the same overall percentage of baskets scored. How many more baskets did Candace score in the second half than in the first?

	First Half	Second Half
Steph	$\frac{15}{20}$	$\frac{10}{10}$
Candace	$\frac{\square}{12}$	$\frac{\square}{18}$

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

🧰 #8 Weights matter #3 Count the range

Key idea: work with totals, keep the halves as inequalities. Steph overall: $\frac{25}{30}$. Candace made x then y baskets: $\frac{x+y}{30} = \frac{25}{30} \Rightarrow x + y = 25$. Per half: $\frac{x}{12} < \frac{15}{20} \Rightarrow x < 9$, and $\frac{y}{18} < \frac{10}{10} \Rightarrow y < 18$. But $x \le 8$ and $y \le 17$ already add to at most 25 — so both must hit their maximum: $x = 8$, $y = 17$, and $y - x = 9$. **Answer: C**

2015 AMC 8 #24 · integer solutions

Example 4. A baseball league consists of two four-team divisions. Each team plays every other team in its division N games. Each team plays every team in the other division M games with $N > 2M$ and $M > 4$. Each team plays a 76 game schedule. How many games does a team play within its own division?

(A) 36 **(B)** 48 **(C)** 54 **(D)** 60 **(E)** 72

🧰 #10 Divisibility #2 Sign rules

Key idea: a team plays 3 division rivals N times and 4 outside teams M times: $3N + 4M = 76$. Substitute $N = \frac{76-4M}{3}$ into $N > 2M$: $76 - 4M > 6M \Rightarrow M < 7.6$; with $M > 4$, $M \in \{5, 6, 7\}$. Only $M = 7$ makes $N = \frac{48}{3} = 16$ an integer (and $16 > 14$ ✓). In-division games: $3N = 48$. **Answer: B**

✏️ Practice

Try each one before opening the solution.

2005 AMC 8 #6

1. Suppose d is a digit. For how many values of d is $2.00d5 > 2.005$?

(A) 0 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 10

2007 AMC 8 #15

2. Let a, b and c be numbers with $0 < a < b < c$. Which of the following is impossible?

(A) $a+c**(B)** $a\cdot b**(C)** $a+b**(D)** $a\cdot c**(E)** $\frac{b}{c} = a$

2005 AMC 8 #15

3. How many different isosceles triangles have integer side lengths and perimeter 23?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 9 **(E)** 11

2018 AMC 10A #4

4. Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least 6 miles away," Bob replied, "We are at most 5 miles away." Charlie then remarked, "Actually the nearest town is at most 4 miles away." It turned out that none of the three statements were true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d?

(A) $(0,4)$ **(B)** $(4,5)$ **(C)** $(4,6)$ **(D)** $(5,6)$ **(E)** $(5,\infty)$

🆕 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2025 AMC 8 #7

5. On the most recent exam in Prof. Xochi's class, 5 students earned a score of at least 95%, 13 students earned a score of at least 90%, 27 students earned a score of at least 85%, and 50 students earned a score of at least 80%. How many students earned a score of at least 80% and less than 90%?

(A) 8 **(B)** 14 **(C)** 22 **(D)** 37 **(E)** 45

2026 AMC 8 #8

6. A poll asked a number of people if they liked solving mathematics problems. Exactly 74% answered "yes." What is the fewest possible number of people who could have been asked the question?

(A) 10 **(B)** 20 **(C)** 25 **(D)** 50 **(E)** 100

2024 AMC 8 #15

7. Let the letters F, L, Y, B, U, G represent distinct digits. Suppose $\underline{F}\,\underline{L}\,\underline{Y}\,\underline{F}\,\underline{L}\,\underline{Y}$ is the greatest number that satisfies the equation

$$8 \cdot \underline{F}\,\underline{L}\,\underline{Y}\,\underline{F}\,\underline{L}\,\underline{Y} = \underline{B}\,\underline{U}\,\underline{G}\,\underline{B}\,\underline{U}\,\underline{G}.$$

What is the value of $\underline{F}\,\underline{L}\,\underline{Y} + \underline{B}\,\underline{U}\,\underline{G}$?

(A) 1089 **(B)** 1098 **(C)** 1107 **(D)** 1116 **(E)** 1125

Sources: prep-course Lecture 11 (video + board notes + 8 worked examples, translated to English) and official AMC 8 problems from the local bank (`kb/content/banks/`). Answers verified against official AoPS answer keys where available.

🗝️ Answer key

Example	1	2	3	4			
	A	D	C	B			
Practice	1	2	3	4	5	6	7
	C	A	C	D	D	D	C

📝 Solutions to practice problems

Practice 1.

🧰 #3 Count the range

Write 2.005 as 2.0050 and compare place by place: the numbers agree until the thousandths digit, d vs 5. If $d > 5$ we win; if $d = 5$, then $2.0055 > 2.0050$ still wins; if $d < 5$ we lose. So $d \in \{5, 6, 7, 8, 9\}$ — five values. **Answer: C**

Practice 2.

🧰 #5 Extreme numbers

Since $c > b$ and $a > 0$, adding a to c only grows it: $a + c > c > b$ — always. So (A) is impossible. The rest all happen: $a = 1, b = 2, c = 10$ kills (B) and (C); $a = \frac{1}{1000}, b = 1, c = 2$ kills (D); $a = \frac{1}{2}, b = 4, c = 8$ kills (E). **Answer: A**

Practice 3.

🧰 #6 Triangle inequality #3 Count the range

Legs a, base b: $2a + b = 23$ and the triangle inequality $2a > b$ give $23 - b > b$, so $b < 11.5$. Also $a = \frac{23-b}{2}$ must be an integer, forcing b odd. Odd values $1 \le b \le 11$: that's $b \in \{1, 3, 5, 7, 9, 11\}$ — six triangles. **Answer: C**

Practice 4.

#3 Intervals #2 Sign rules

Negate each claim: Alice false $\Rightarrow d < 6$; Bob false $\Rightarrow d > 5$; Charlie false $\Rightarrow d > 4$. Intersect: $5 < d < 6$, the open interval $(5, 6)$. **Answer: D**

Practice 5.

#3 Intervals #9 Subtract ranges

"At least 80 and less than 90" is everyone at ≥ 80 minus everyone at ≥ 90 — the "at least" groups nest inside each other. $50 - 13 = 37$. **Answer: D**

Practice 6.

#10 Divisibility

With n people, the "yes" count is $\frac{74}{100}n = \frac{37}{50}n$, and it must be a whole number. Since 37 and 50 share no factor, n must be a multiple of 50 — fewest is 50 (then 37 people said yes). **Answer: D**

Practice 7.

#9 Bound it #1 Balance moves

$\underline{F}\,\underline{L}\,\underline{Y}\,\underline{F}\,\underline{L}\,\underline{Y} = \overline{FLY} \times 1001$, so dividing both sides by 1001 shrinks the equation to $8 \cdot \overline{FLY} = \overline{BUG}$. Bound it: $\overline{BUG} \leq 999 \Rightarrow \overline{FLY} \leq 124$. Test from the top: $124 \to 992$ repeats digits; $123 \to 984$ gives six distinct digits $1, 2, 3, 9, 8, 4$ ✓. $123 + 984 = 1107$. **Answer: C**