Powers & Radicals · Lecture 08 · `alg-powers-radicals`

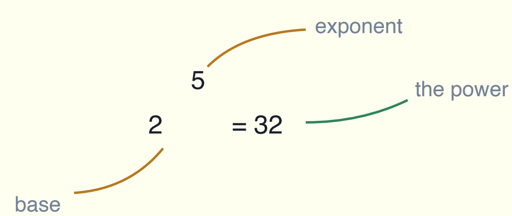
📺 **Course materials — Lecture 08:** lecture video · handout PDF (Lecture 8) · board notes (08.jpg) · homework answer key (Chapter 8) — all in the `amc8/video` folder. · 🖨️ **Printable PDF** (answers & solutions at the back)

🔑 Key facts

A power a^n is n copies of a multiplied together, and two laws run the whole chapter: $a^m \cdot a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$. A radical $\sqrt{x}$ is the **nonnegative** number whose square is x — powers and roots undo each other. Nearly every AMC 8 problem here is won by rewriting everything with the *same base* or the *same exponent*.

🧰 Instructor's toolbox (from the lecture video & board notes)

1 · Name the parts, read the meaning. In $2^5 = 32$: the 2 is the **base**, the 5 is the **exponent**, and the whole thing is the **power**. The board's first move is always to un-abbreviate: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$. When a power problem stalls, write it out — patterns (and cancellations) appear instantly. Know the small ones cold: powers of 2 up to $2^{10} = 1024$ and perfect squares up to $20^2 = 400$.



2 · The two exponent laws that do everything. $a^m \cdot a^n = a^{m+n}$ (same base: add exponents) and $(a^m)^n = a^{mn}$ (power of a power: multiply). That's how 100^{2x} becomes 10^{4x} in Practice 4, how $2^{18} = (2^6)^3 = 64^3$ turns a huge range into a cube count in Practice 7, and how every comparison in Example 3 is set up.

3 · Isolate the power first, then match bases. Before any exponent trick, get the power alone on one side: $3^p + 3^4 = 90 \Rightarrow 3^p = 90 - 81 = 9 = 3^2$, so $p = 2$ — then read the exponent off matching bases (Example 1). Same move three times: $2^r = 76 - 44 = 32 = 2^5$, $6^s = 1421 - 125 = 1296 = 6^4$.

4 · Prime factorization reads off exponents. Factorizations are *unique*, so if $2^w \cdot 3^x \cdot 5^y \cdot 7^z = 588 = 2^2 \cdot 3 \cdot 7^2$, then w, x, y, z are forced: $2, 1, 0, 2$ — a missing prime means exponent 0, not "no answer" (Example 2). Factor trees are the door into every "what power of … divides …" question.

5 · Compare big powers with a common exponent. You can't eyeball 10^8 vs 5^{12} vs 2^{24} — so give them the same exponent, $\gcd(8, 12, 24) = 4$:

$$10^8 = (10^2)^4 = 100^4, \qquad 5^{12} = (5^3)^4 = 125^4, \qquad 2^{24} = (2^6)^4 = 64^4,$$

and now just compare the bases $64 < 100 < 125$ (Example 3). Same exponent → compare bases; same base → compare exponents.

6 · $\sqrt{9} = 3$, never ± 3. The radical sign means the *principal* (nonnegative) root — the board underlines " $\sqrt{9} = 3$ or -3" as the classic error. But the *equation* $x^2 = 9$ really does have two roots, $x = \pm 3$ — that's where the $\pm$ lives. Consequence: $\sqrt{a^2} = |a|$, so $\sqrt{(3 - 2\sqrt{3})^2} = |3 - 2\sqrt{3}| = 2\sqrt{3} - 3$, because $2\sqrt{3} \approx 3.46 > 3$ (Practice 5).

7 · Radical survival kit. $\sqrt{A \cdot B} = \sqrt{A} \cdot \sqrt{B}$ and $\sqrt{\frac{A}{B}} = \frac{\sqrt{A}}{\sqrt{B}}$ (for $A, B \geq 0$, $B \neq 0$). Simplify by pulling out the *perfect square* factor: $\sqrt{40} = \sqrt{4 \times 10} = 2\sqrt{10}$. Clear a root from a denominator by multiplying by it: $\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$ — both straight from the board.

8 · Nested radicals unwind from the inside. For $\sqrt{16\sqrt{8\sqrt{4}}}$, start at the deepest layer and work out: $\sqrt{4} = 2$, then $\sqrt{8 \cdot 2} = \sqrt{16} = 4$, then $\sqrt{16 \cdot 4} = \sqrt{64} = 8$ (Example 4). Never try to distribute the outer root across the inside.

9 · Conjugates and $a^2 - b^2 = (a + b)(a - b)$. The difference of squares runs both directions here: it factors monsters like $13^4 - 11^4 = (13^2 + 11^2)(13^2 - 11^2)$ (Practice 6) and $k^2 - 1 = (k - 1)(k + 1)$ (Fresh 10), and it collapses radical conjugates: $(9 - 6\sqrt{2})(9 + 6\sqrt{2}) = 81 - 72 = 9$. When a *sum* of ugly radicals is asked for, square the whole expression, simplify, then take the root back (Challenge C2).

10 · Estimate: round, then shift decimal points in pairs. $0.000315 \times 7{,}928{,}564 \approx 0.0003 \times 8{,}000{,}000$ — now move one factor up by 1000 and the other down by 1000: $0.3 \times 8000 = 2400$ (Practice 1). Multiplying and dividing by the same power of 10 keeps the product honest and the zeros manageable.

📖 Worked examples

2013 AMC 8 #15 · match the base

Example 1. If $3^p + 3^4 = 90$, $2^r + 44 = 76$, and $5^3 + 6^s = 1421$, what is the product of p, r, and s?

(A) 27 **(B)** 40 **(C)** 50 **(D)** 70 **(E)** 90

🧰 #3 Isolate the power #1 Know small powers

Key idea: isolate each power, then write both sides with the same base. $3^p = 90 - 81 = 9 = 3^2 \Rightarrow p = 2$. $2^r = 76 - 44 = 32 = 2^5 \Rightarrow r = 5$. $6^s = 1421 - 125 = 1296 = 6^4 \Rightarrow s = 4$. Product: $2 \cdot 5 \cdot 4 = 40$. **Answer: B**

2011 AMC 8 #17 · prime factorization

Example 2. Let w, x, y, and z be whole numbers. If $2^w \cdot 3^x \cdot 5^y \cdot 7^z = 588$, then what does $2w + 3x + 5y + 7z$ equal?

(A) 21 **(B)** 25 **(C)** 27 **(D)** 35 **(E)** 56

🧰 #4 Prime factorization

Key idea: factor 588 into primes: $588 = 4 \cdot 147 = 2^2 \cdot 3 \cdot 7^2$. Prime factorization is unique, so $w = 2$, $x = 1$, $y = 0$ (no factor of 5!), $z = 2$. Then $2w + 3x + 5y + 7z = 4 + 3 + 0 + 14 = 21$. **Answer: A**

2010 AMC 8 #24 · comparing powers

Example 3. What is the correct ordering of the three numbers 10^8, 5^{12}, and 2^{24}?

(A) $2^{24} < 10^8 < 5^{12}$ **(B)** $2^{24} < 5^{12} < 10^8$ **(C)** $5^{12} < 2^{24} < 10^8$ **(D)** $10^8 < 5^{12} < 2^{24}$ **(E)** $10^8 < 2^{24} < 5^{12}$

🧰 #5 Common exponent #2 Exponent laws

Key idea: the exponents $8, 12, 24$ share the factor 4 — rewrite everything as a fourth power: $10^8 = (10^2)^4 = 100^4, \quad 5^{12} = (5^3)^4 = 125^4, \quad 2^{24} = (2^6)^4 = 64^4$. Same exponent, so compare bases: $64 < 100 < 125$, giving $2^{24} < 10^8 < 5^{12}$. **Answer: A**

2017 AMC 8 #3 · nested radicals

Example 4. What is the value of the expression $\sqrt{16\sqrt{8\sqrt{4}}}$?

(A) 4 **(B)** $4\sqrt{2}$ **(C)** 8 **(D)** $8\sqrt{2}$ **(E)** 16

#8 Inside out #7 Radical kit

Key idea: unwind from the deepest radical: $\sqrt{4} = 2$, so the middle layer is $\sqrt{8 \cdot 2} = \sqrt{16} = 4$, and the outside is $\sqrt{16 \cdot 4} = \sqrt{64} = 8$. **Answer: C**

✏️ Practice

Try each one before opening the solution.

2017 AMC 8 #4

1. When 0.000315 is multiplied by $7{,}928{,}564$ the product is closest to which of the following?

(A) 210 **(B)** 240 **(C)** 2100 **(D)** 2400 **(E)** 24000

2022 AMC 8 #2

2. Consider these two operations:

$$a \blacklozenge b = a^2 - b^2 \qquad\qquad a \bigstar b = (a-b)^2$$

What is the value of $(5 \blacklozenge 3) \bigstar 6$?

(A) -20 **(B)** 4 **(C)** 16 **(D)** 100 **(E)** 220

2016 AMC 8 #7

3. Which of the following numbers is **not** a perfect square?

(A) 1^{2016} **(B)** 2^{2017} **(C)** 3^{2018} **(D)** 4^{2019} **(E)** 5^{2020}

2016 AMC 10A #2

4. For what value of x does $10^x \cdot 100^{2x} = 1000^5$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2021 AMC 10B #2

5. What is the value of $\sqrt{(3-2\sqrt{3})^2}+\sqrt{(3+2\sqrt{3})^2}$?

(A) 0 **(B)** $4\sqrt{3}-6$ **(C)** 6 **(D)** $4\sqrt{3}$ **(E)** $4\sqrt{3}+6$

2016 AMC 8 #15

6. What is the largest power of 2 that is a divisor of 13^4-11^4?

(A) 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 128

2018 AMC 8 #25

7. How many perfect cubes lie between 2^8+1 and $2^{18}+1$, inclusive?

(A) 4 **(B)** 9 **(C)** 10 **(D)** 57 **(E)** 58

🆕 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2024 AMC 8 #4

8. When Yunji added all the integers from 1 to 9, she mistakenly left out a number. Her incorrect sum turned out to be a square number. What number did Yunji leave out?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

2026 AMC 8 #9

9. What is the value of this expression?

$$\frac{\sqrt{16\sqrt{81}}}{\sqrt{81\sqrt{16}}}$$

(A) $\frac{4}{9}$ **(B)** $\frac{2}{3}$ **(C)** 1 **(D)** $\frac{3}{2}$ **(E)** $\frac{9}{4}$

2025 AMC 8 #23

10. How many four-digit numbers have all three of the following properties? (I) The tens and ones digit are both 9. (II) The number is 1 less than a perfect square. (III) The number is the product of exactly two prime numbers.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

⭐ Challenge (AMC 10 level)

2003 AMC 10B #9 · exponent equation

C1. Find the value of x that satisfies the equation

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}.$$

(A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 9

course problem · radical conjugates

C2. Which of the following is equal to $\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}$?

(A) $3\sqrt{2}$ **(B)** $2\sqrt{6}$ **(C)** $\frac{7\sqrt{2}}{2}$ **(D)** $3\sqrt{3}$ **(E)** 6

Sources: prep-course Lecture 08 (video + board notes + 9 worked examples, translated to English) and official AMC 8 problems from the local bank (`kb/content/banks/`), which also supplied the extra on-topic practice (2016 #7, 2016 #15, 2018 #25, 2022 #2). Answers cross-checked between the course solutions and the bank's official answer keys.

🗝️ Answer key

Example	1	2	3	4						
	B	A	A	C						
Practice	1	2	3	4	5	6	7	8	9	10
	D	D	B	C	D	C	E	E	B	B
Challenge	C1	C2								
	B	B								

📝 Solutions to practice problems

Practice 1.

🧰 #10 Round and shift

Round: $0.0003 \times 8{,}000{,}000$. Shift by 1000 both ways: $0.3 \times 8000 = 2400$. **Answer: D**

Practice 2.

🧰 #9 Difference of squares

Inside first: $5 \blacklozenge 3 = 5^2 - 3^2 = 25 - 9 = 16$. Then $16 \bigstar 6 = (16 - 6)^2 = 10^2 = 100$. ($a^2 - b^2$ and $(a-b)^2$ are *not* the same thing — that's the whole point of the problem.) **Answer: D**

Practice 3.

🧰 #2 Exponent laws #4 Prime factorization

A power is a perfect square when it can be written as $(\text{something})^2$ — i.e. its exponent is even (over a prime base). Even exponents: A, C, E ✓. And $4^{2019} = (2^2)^{2019} = 2^{4038}$ — even too ✓. Only 2^{2017} has an odd exponent of a prime. **Answer: B**

Practice 4.

#2 Exponent laws #3 Match bases

Everything is a power of 10: $10^x \cdot (10^2)^{2x} = 10^x \cdot 10^{4x} = 10^{5x}$ and $1000^5 = (10^3)^5 = 10^{15}$. So $5x = 15$, $x = 3$. **Answer: C**

Practice 5.

#6 Principal root

$\sqrt{a^2} = |a|$, so the expression is $|3 - 2\sqrt{3}| + |3 + 2\sqrt{3}|$. Since $2\sqrt{3} \approx 3.46 > 3$, the first term is $2\sqrt{3} - 3$ (not $3 - 2\sqrt{3}$!), the second is $3 + 2\sqrt{3}$. Sum: $(2\sqrt{3} - 3) + (2\sqrt{3} + 3) = 4\sqrt{3}$. **Answer: D**

Practice 6.

#9 Difference of squares #4 Prime factorization

Factor twice: $13^4 - 11^4 = (13^2 + 11^2)(13^2 - 11^2) = (169 + 121)(169 - 121) = 290 \cdot 48$. Count the 2s: $290 = 2 \cdot 145$ and $48 = 2^4 \cdot 3$, so the product has $2^5 = 32$. **Answer: C**

Practice 7.

#2 Exponent laws #1 Know small powers

$2^8 + 1 = 257$: since $6^3 = 216 < 257 \leq 343 = 7^3$, the smallest cube in range is 7^3. $2^{18} = (2^6)^3 = 64^3$, and $64^3 \leq 2^{18} + 1$, so the largest is 64^3. Cubes of $7, 8, \ldots, 64$: that's $64 - 7 + 1 = 58$. **Answer: E**

Practice 8.

#1 Know small powers

$1 + 2 + \cdots + 9 = 45$. Leaving out a number from 1 to 9 puts the sum between 36 and 44 — the only perfect square there is 36, so she left out $45 - 36 = 9$. **Answer: E**

Practice 9.

#8 Inside out · #7 Radical kit

Inside out, top and bottom: numerator $\sqrt{16 \cdot 9} = \sqrt{144} = 12$; denominator $\sqrt{81 \cdot 4} = \sqrt{324} = 18$. So the value is $\frac{12}{18} = \frac{2}{3}$. **Answer: B**

Practice 10.

#9 Difference of squares · #1 Know small powers

Write the number as $k^2 - 1 = (k-1)(k+1)$. It ends in 99, so k^2 ends in 00 — k is a multiple of 10, and four digits force $k \in \{40, 50, \dots, 100\}$. For exactly two prime factors we need $k-1$ and $k+1$ *both prime*: $39, 49, 69, 81, 91, 99$ all fail; only $k = 60$ works (59 and 61 are twin primes), giving $3599 = 59 \cdot 61$. Exactly one such number. **Answer: B**

Practice C1.

#3 Match bases · #2 Exponent laws

Everything in base 5. Left: $25^{-2} = 5^{-4}$. Right: $25^{17/x} = 5^{34/x}$, so the exponent is $\frac{48}{x} - \frac{26}{x} - \frac{34}{x} = \frac{-12}{x}$. Match exponents: $-4 = \frac{-12}{x} \Rightarrow x = 3$. **Answer: B**

Practice C2.

#9 Conjugates · #7 Radical kit

Call the whole thing S (note $S > 0$) and square it:

$$S^2 = (9 - 6\sqrt{2}) + (9 + 6\sqrt{2}) + 2\sqrt{(9 - 6\sqrt{2})(9 + 6\sqrt{2})} = 18 + 2\sqrt{81 - 72} = 18 + 2 \cdot 3 = 24.$$

So $S = \sqrt{24} = 2\sqrt{6}$. **Answer: B**