Sequences & Series · Lecture 10 · `alg-sequences-series`

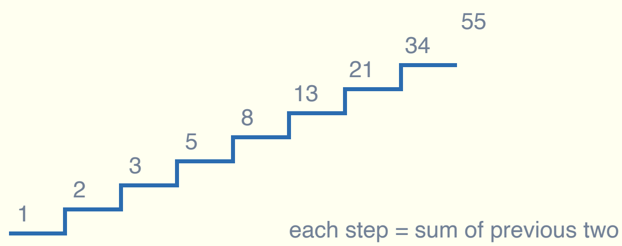
📼 **Course materials — Lecture 10 (92 min):** lecture video · handout PDF (Lecture 10) · board notes (10.jpg) · homework answer key (Chapter 10) — all in the `amc8/video` folder. · 🖨️ **Printable PDF** (answers & solutions at the back)

🔑 Key facts

An **arithmetic sequence** adds the same number d each step: $a_n = a_1 + (n-1)d$. A **geometric sequence** multiplies by the same ratio q each step: $a_n = a_1 \cdot q^{n-1}$. Almost every AMC 8 sequence problem is won with three little formulas: number of terms $= \dfrac{\text{last} - \text{first}}{d} + 1$, sum $= \dfrac{(\text{first} + \text{last}) \times n}{2}$, and "each term is the **average of its two neighbors**" (arithmetic).

🧰 Instructor's toolbox (from the lecture video & board notes)

1 · Know the three families on sight. Arithmetic $1, 2, 3, 4, 5, \ldots$ (add d), geometric $1, 2, 4, 8, 16, \ldots$ (multiply by q), and Fibonacci $1, 2, 3, 5, 8, \ldots$ (each term = sum of the previous two — the "rabbit sequence"). The instructor's staircase: to climb 9 steps taking 1 or 2 at a time, the counts per step are Fibonacci — $1, 2, 3, 5, 8, 13, 21, 34, \mathbf{55}$ ways.



2 · The difference must be the same at EVERY gap. $d = \text{later term} - \text{earlier term}$: for $1, 3, 5, 7, 9, \ldots$ $d = 3 - 1 = 2$; for $99, 97, 95, \ldots$ $d = 97 - 99 = -2$ (counting down is allowed). The instructor's trap: $1, 2, 1, 2, 1, 2, \ldots$ is **not** arithmetic — the gaps alternate $+1, -1$. Constant sequences like $1, 1, 1, 1, \ldots$ *are* arithmetic ($d = 0$).

3 · Jump between distant terms in one hop. $a_n = a_m + (n-m)\,d$ — from a_2 to a_{10} is 8 jumps, not 9. Board examples: $a_2 = 4,\ a_{10} = 44 \Rightarrow d = \frac{44-4}{10-2} = 5$; and $a_5 = 12,\ a_{10} = 47 \Rightarrow d = \frac{47-12}{5} = 7$. Bonus: each term is the average of its neighbors, $b = \frac{a+c}{2}$ — that one line cracks Example 3 and Fresh 7.

4 · Count terms like fence posts. $n = \dfrac{\text{last} - \text{first}}{d} + 1$ — the "$+1$" is the tree-planting fence-post correction, the instructor's most-repeated warning. Board example: $2, 5, 8, \ldots, 47$ has $\frac{47-2}{3} + 1 = 16$ terms. Cross-check by shifting: add 1 to every term ($3, 6, \ldots, 48$), divide by 3 ($1, 2, \ldots, 16$) — sixteen terms, no formula needed.

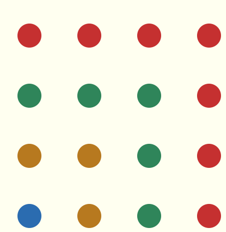
5 · Sum an arithmetic series by pairing the ends. $S = \dfrac{(\text{first} + \text{last}) \times n}{2}$. Board examples: $1 + 3 + 5 + \cdots + 39 = \frac{(1+39)\times 20}{2} = 400$ and $90 + 91 + \cdots + 99 = \frac{(90+99)\times 10}{2} = 945$. Write the sum forwards and backwards and every column adds to the same number — that's the whole proof.

20 terms → 40 × 20 ÷ 2 = 400

1	3	5	…	39	each pair = 40
|	|	|		|	
39	37	35	…	1	

6 · Consecutive odd numbers from 1: the sum is a perfect square. $1 + 3 + 5 + \cdots + (2k-1) = k^2$ — the instructor's slogan: "one of a kind: *square the number of terms*." So $1 + 3 + 5 + \cdots + 99 = 50^2 = 2500$, instantly. Picture: each new odd number is an L-shaped shell that grows the dot-square by one row and one column. This is exactly why the can display in Practice 2 is n^2.

1 + 3 + 5 + 7 = 4^2



7 · Geometric jumps: multiply by q once per step. $a_n = a_m \cdot q^{\,n-m}$. Board examples: $a_2 = 4,\ a_5 = 108 \Rightarrow q^3 = 27 \Rightarrow q = 3$; and $a_1 = 5,\ q = 5 \Rightarrow a_{10} = 5 \cdot 5^9 = 5^{10}$ (leave it as a power — don't multiply it out). Spot a geometric sequence by *dividing* neighbors: $\frac{a_{n+1}}{a_n}$ must be the same constant every time.

8 · Sum a geometric series: multiply, shift, subtract. The instructor's "misaligned subtraction": write S, write qS under it shifted one slot, subtract — everything in the middle cancels. $S = 1 + 2 + 4 + \cdots + 1024$: then $2S = 2 + 4 + \cdots + 2048$, so $2S - S = 2048 - 1 = 2047$. $S = 1 + 3 + 9 + \cdots + 2187$: then $3S - S = 6561 - 1$, so $S = \frac{6560}{2} = 3280$. (Practice drill from the board: try $4^1 + 4^2 + \cdots + 4^{10}$ the same way.)

📖 Worked examples

2015 AMC 8 #9 · arithmetic series

Example 1. On her first day of work, Janabel sold one widget. On day two, she sold three widgets. On day three, she sold five widgets, and on each succeeding day, she sold two more widgets than she had sold on the previous day. How many widgets in total had Janabel sold after working 20 days?

(A) 39 **(B)** 40 **(C)** 210 **(D)** 400 **(E)** 401

🧰 #3 Jump formula #5 Pair the ends #6 Odd sums

Key idea: daily sales $1, 3, 5, \ldots$ form an arithmetic sequence with $a_1 = 1$, $d = 2$. Day 20: $a_{20} = 1 + 19 \cdot 2 = 39$. Total: $S_{20} = \frac{(1+39)\times 20}{2} = 400$. **Answer: D**
Shortcut: the sum of the first 20 odd numbers is $20^2 = 400$ — no formula needed.

2022 AMC 8 #11 · count the steps

Example 2. Henry the donkey has a very long piece of pasta. He takes a number of bites of pasta, each time eating 3 inches of pasta from the middle of one piece. In the end, he has 10 pieces of pasta whose total length is 17 inches. How long, in inches, was the piece of pasta he started with?

(A) 34 **(B)** 38 **(C)** 41 **(D)** 44 **(E)** 47

🧰 #4 Fence posts

Key idea: every bite from the middle of a piece turns 1 piece into 2 — the piece count grows by exactly 1 per bite. From 1 piece to 10 pieces takes 9 bites (not 10 — fence posts!), eating $9 \times 3 = 27$ inches. Original length $= 27 + 17 = 44$ inches. **Answer: D**

2015 AMC 8 #18 · middle term = average

Example 3. An arithmetic sequence is a sequence in which each term after the first is obtained by adding a constant to the previous term. For example, $2, 5, 8, 11, 14$ is an arithmetic sequence with five terms, in which the first term is 2 and the constant added is 3. Each row and each column in this 5×5 array is an arithmetic sequence with five terms. What is the value of X?

1				25
		X		
17				81

(A) 21 **(B)** 31 **(C)** 36 **(D)** 40 **(E)** 42

#3 Middle = average

Key idea: in a 5-term arithmetic sequence the middle term is the average of the two ends. Top row: middle $A = \frac{1+25}{2} = 13$. Bottom row: middle $B = \frac{17+81}{2} = 49$. The middle *column* is also arithmetic, and X is its center: $X = \frac{A+B}{2} = \frac{13+49}{2} = 31$. **Answer: B**

2013 AMC 8 #9 · geometric growth

Example 4. The Incredible Hulk can double the distance he jumps with each succeeding jump. If his first jump is 1 meter, the second jump is 2 meters, the third jump is 4 meters, and so on, then on which jump will he first be able to jump more than 1 kilometer?

(A) 9th **(B)** 10th **(C)** 11th **(D)** 12th **(E)** 13th

🧰 #7 Geometric jumps

Key idea: jump lengths are geometric with $a_1 = 1, q = 2$: the n-th jump is 2^{n-1} meters. We need $2^{n-1} > 1000$. Since $2^9 = 512 < 1000$ and $2^{10} = 1024 > 1000$, we need $n - 1 = 10$, i.e. the 11th jump. (Memorize $2^{10} = 1024 \approx 1000$ — it shows up constantly.) **Answer: C**

✏️ Practice

Try each one before opening the solution. Figures are from the official exams.

mock · course problem

1. $90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 = ?$

(A) 845 **(B)** 945 **(C)** 1005 **(D)** 1025 **(E)** 1045

2004 AMC 10B #10

2. A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

(A) 5 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

2008 AMC 8 #12

3. A ball is dropped from a height of 3 meters. On its first bounce it rises to a height of 2 meters. It keeps falling and bouncing to $\frac{2}{3}$ of the height it reached in the previous bounce. On which bounce will it rise to a height less than 0.5 meters?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

2003 AMC 10B #8

4. The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?

(A) $-\sqrt{3}$ **(B)** $-\frac{2\sqrt{3}}{3}$ **(C)** $-\frac{\sqrt{3}}{3}$ **(D)** $\sqrt{3}$ **(E)** 3

mock · course problem

5. In a geometric sequence $6, \ldots, 768, \ldots, 12288$, the term 768 is the n-th term and 12288 is the $(2n-4)$-th term. Find the common ratio q.

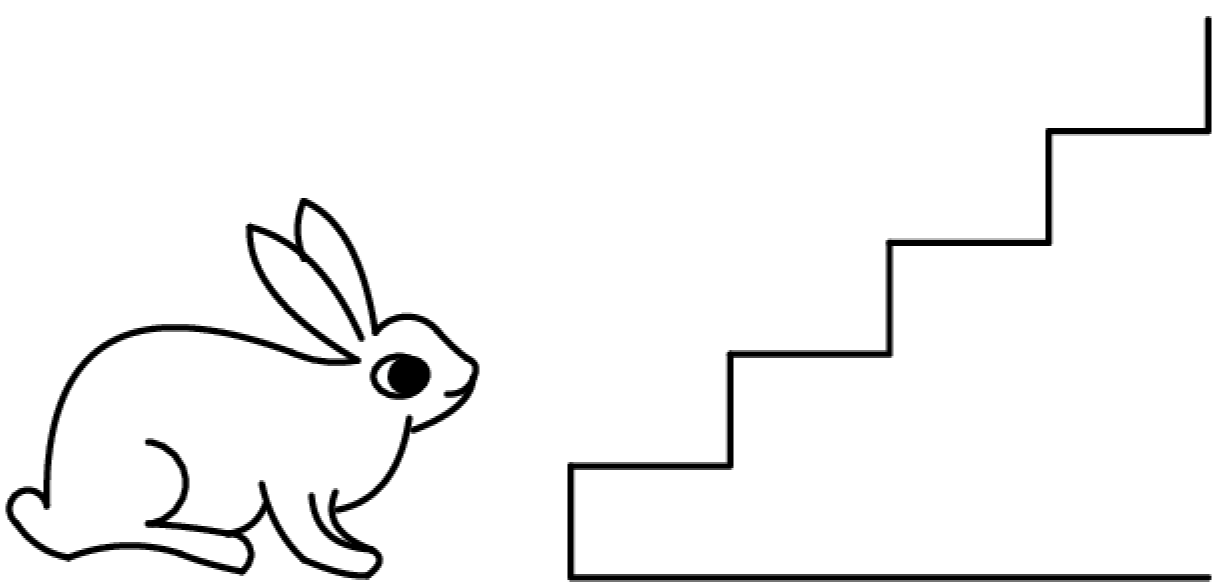
(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 8

🆕 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2024 AMC 8 #13

6. Buzz Bunny is hopping up and down a set of stairs, one step at a time. In how many ways can Buzz Bunny start on the ground, make a sequence of 6 hops, and end up back on the ground? (For example, one sequence of hops is up-up-down-down-up-down.)



(A) 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 12

2026 AMC 8 #14

7. Jami picked three equally spaced integer numbers on the number line. The sum of the first and the second numbers is 40, while the sum of the second and third numbers is 60. What is the sum of all three numbers?

(A) 70 **(B)** 75 **(C)** 80 **(D)** 85 **(E)** 90

2026 AMC 8 #18

8. In how many ways can 60 be written as the sum of two or more consecutive odd positive integers that are arranged in increasing order?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

⭐ Challenge (AMC 10 level)

2002 AMC 10B #19 · arithmetic series

C1. Suppose that $\{a_n\}$ is an arithmetic sequence with $a_1 + a_2 + \cdots + a_{100} = 100$ and $a_{101} + a_{102} + \cdots + a_{200} = 200$. What is the value of $a_2 - a_1$?

(A) 0.0001 **(B)** 0.001 **(C)** 0.01 **(D)** 0.1 **(E)** 1

Sources: prep-course Lecture 10 (video + board notes + 10 worked examples, translated to English) and official AMC 8 problems from the local bank (`kb/content/banks/`). Answers verified against official AoPS answer keys where available.

🗝️ Answer key

Example	1	2	3	4				
	D	D	B	C				
Practice	1	2	3	4	5	6	7	8
	B	D	C	B	A	B	B	B
Challenge	C1							
	C							

📝 Solutions to practice problems

Practice 1.

🧰 #5 Pair the ends

Ten terms, first 90, last 99: $S = \frac{(90+99)\times 10}{2} = \frac{1890}{2} = 945$. **Answer: B**

Practice 2.

🧰 #6 Odd sums #5 Pair the ends

Rows hold $1, 3, 5, \ldots$ cans — consecutive odd numbers starting at 1, so n rows hold n^2 cans. $n^2 = 100 \Rightarrow n = 10$. (Or solve $\frac{(1+(2n-1))\,n}{2} = 100$.) **Answer: D**

Practice 3.

🧰 #7 Geometric jumps

Bounce heights are geometric: $a_n = 2\left(\frac{2}{3}\right)^{n-1}$. Track them: $2,\ \frac{4}{3},\ \frac{8}{9},\ \frac{16}{27},\ \frac{32}{81}$. Since $\frac{16}{27} \approx 0.59 > 0.5$ and $\frac{32}{81} \approx 0.40 < 0.5$, the 5th bounce is the first below half a meter. **Answer: C**

Practice 4.

🧰 #7 Geometric jumps

$a_4 = a_2 \cdot q^2$, so $q^2 = \frac{6}{2} = 3$ and $q = \pm\sqrt{3}$. Then $a_1 = \frac{a_2}{q} = \frac{2}{\pm\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$ — and the minus version is choice B. **Answer: B**

Practice 5.

🧰 #7 Geometric jumps

$768 = 6q^{n-1}$ and $12288 = 6q^{2n-5}$. Divide: $\frac{12288}{768} = 16 = q^{n-4}$. Also $\frac{768}{6} = 128 = q^{n-1}$. Try $q = 2$: $2^{n-1} = 128 \Rightarrow n = 8$, and then $q^{n-4} = 2^4 = 16$ ✓. **Answer: A**

Practice 6.

🧰 #1 Staircase counting

List systematically like the lecture's staircase: 3 hops are U and 3 are D, and the bunny can never dip below the ground (every prefix needs at least as many U's as D's). The valid orders are UUUDDD, UUDUDD, UUDDUD, UDUUDD, UDUDUD — 5 ways. **Answer: B**

Practice 7.

🧰 #3 Middle = average

Equally spaced means arithmetic: call the numbers $m-d,\ m,\ m+d$. Then $(m-d)+m = 40$ and $m + (m+d) = 60$; adding the two equations gives $4m = 100$, so $m = 25$. The total is $3m = 75$ (the middle term is the average, so the sum is $3\times$ the middle). **Answer: B**

Practice 8.

🧰 #5 Pair the ends #4 Fence posts

n consecutive odd numbers starting at a: the last is $a + 2(n-1)$, so the sum is $\frac{(a+a+2(n-1))\,n}{2} = n(a+n-1) = 60$. So n divides 60 and $a = \frac{60}{n} - n + 1$ must be a *positive odd* number. Check divisors $n \geq 2$: $n = 2 \Rightarrow a = 29$ ✓ $(29+31)$; $n = 3, 4, 5 \Rightarrow a = 18, 12, 8$ — even ✗; $n = 6 \Rightarrow a = 5$ ✓ $(5+7+9+11+13+15)$; $n = 10 \Rightarrow a < 0$ ✗. Two ways. **Answer: B**

Practice C1.

🧰 #3 Jump formula

Subtract the two sums term by term: $a_{101} - a_1 = 100d$, $a_{102} - a_2 = 100d$, …, one hundred pairs, each equal to $100d$. So $100 \cdot 100d = 200 - 100 = 100$, giving $d = 0.01$ — and $a_2 - a_1 = d$. **Answer: C**