

Permutations & Combinations · Lecture 12 · cp-permutations-combinations

 **Course materials — Lecture 12:** lecture video · handout PDF (Lecture 12) · board notes (12.jpg) · homework answer key (Chapter 12) — all in the `amc8/video` folder. ·  **Printable PDF** (answers & solutions at the back)

Key facts

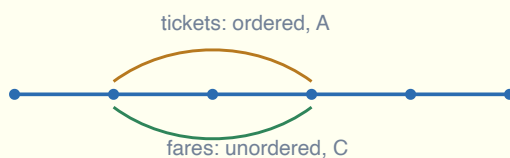
Multiplication rule: a task done in steps → multiply the counts of each step. **Addition rule:** a task split into non-overlapping cases → add the counts. **Permutations** (order matters): $A_n^k = n(n-1) \cdots (n-k+1)$, and $A_n^n = n!$. **Combinations** (order doesn't matter): $C_n^k = \frac{A_n^k}{k!}$ — every unordered choice was counted $k!$ times. The first question to ask on every counting problem: *does order matter?*

Instructor's toolbox (from the lecture video & board notes)

1 · Multiplication rule — steps, not choices. Break the task into steps that *all* must happen: an outfit = shirt **and** skirt **and** shoes → $5 \cdot 4 \cdot 3 = 60$. The instructor's warning: multiply only when every step is required — if any step can be skipped, it isn't one task in steps, it's separate cases (use tip #2).

2 · Addition rule — split into clean cases. When one recipe doesn't cover everything, cut the problem into cases that can't overlap and add. Classic split: casework on the last digit (Example 2: units digit 0 or 5), on parity, or on "how many rows of triangles" (Practice 14). Always check: no overlaps, nothing missed.

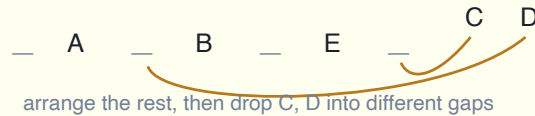
3 · A or C? Ask "does order matter?" Picking 2 of $\{1, 2, 3\}$: ordered gives 12, 21, 13, 31, 23, 32 — that's $A_3^2 = 6$; as mere selections only $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ remain — $C_3^2 = 3$. The instructor's rail-line test: 6 stations on a line → one-way *tickets* are ordered, $A_6^2 = 30$; ticket *prices* are unordered (a round trip costs the same), $C_6^2 = 15$.



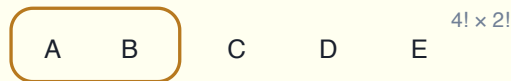
4 · Combination identities — compute the small side. $C_m^n = C_m^{m-n}$ (choosing who's *in* = choosing who's *out*), so $C_{10}^8 = C_{10}^2 = 45$. Also $C_n^n = C_n^0 = 1$. Never grind out C_{10}^8 the long way.

5 · Complement — count the opposite. When the direct count splinters into many cases, count what you *don't* want and subtract from the total. Example 1: passwords not starting 9, 1, 1 $\rightarrow 10^4 - 10$. The board shows the classic wrong answer $9 \cdot 9 \cdot 9 \cdot 10$ (that forbids each digit separately — not the same thing!). Same move cracks Practice 13 (C_6^3 minus collinear triples).

6 · Gap method (for "NOT adjacent"). Arrange the unrestricted items first, then slot the restricted ones into the gaps (ends count!). 5 people, C and D apart: arrange A, B, E ($3! = 6$), they create 4 gaps, place C, D in different gaps in order: $A_4^2 = 12 \rightarrow 6 \cdot 12 = 72$.



7 · Bundling (for "MUST stay together"). Tie the together-items into one super-item, arrange, then remember the orders *inside* the bundle. 2 books must sit together among 5: bundle them ($2!$ inside), arrange 4 items ($4!$): $2 \cdot 24 = 48$.



8 · Repeated items — divide out the copies. Identical items can't be told apart, so divide by $k!$ for each group of k copies: digits 1, 1, 2 $\rightarrow \frac{3!}{2!} = 3$ numbers; INDIANA $\rightarrow \frac{7!}{2! 2! 2!} = 630$; six digits 1, 1, 1, 2, 2, 3 $\rightarrow \frac{6!}{3! 2!}$. Distributing 7 books, 2-2-3, works the same way: $\frac{7!}{2! 2! 3!} = 210$.

9 · Fix relative order by symmetry. If some items' mutual order is *forced*, arrange everything then divide by the orderings of that group. "A left of B" in a row of 5: $\frac{5!}{2!} = 60$. The board's harder one — "A between B and C (not necessarily adjacent)": $\frac{5!}{3!} \times 2 = 40$, since of the $3! = 6$ orders of $\{A, B, C\}$ exactly 2 put A in the middle.

Worked examples

2016 AMC 8 #17 · complement

Example 1. An ATM password at Fred's Bank is composed of four digits from 0 to 9, with repeated digits allowable. If no password may begin with the sequence 9, 1, 1, then how many passwords are possible?

- (A) 30 (B) 7290 (C) 9000 (D) 9990 (E) 9999



#5 Complement

#1 Multiplication rule

Key idea: "does NOT begin with 911" is one clean complement, not four separate digit conditions. Total passwords: $10 \cdot 10 \cdot 10 \cdot 10 = 10000$. Bad ones: 9, 1, 1 fixed, last digit free $\rightarrow 10$. So $10000 - 10 = 9990$. **Answer: D**

The trap: $9 \cdot 9 \cdot 9 \cdot 10 = 7290$ forbids *each* of the first three digits separately — that's a different (wrong) condition.

2011 AMC 8 #23 · casework

Example 2. How many 4-digit positive integers have four different digits, where the leading digit is not zero, the integer is a multiple of 5, and 5 is the largest digit?

- (A) 24 (B) 48 (C) 60 (D) 84 (E) 108



#2 Addition rule

#1 Multiplication rule

Key idea: a multiple of 5 ends in 0 or 5 — perfect casework. **Units = 5:** the other three digits come from $\{0, 1, 2, 3, 4\}$: leading digit 4 ways (not 0), then 4, then 3 $\rightarrow 4 \cdot 4 \cdot 3 = 48$. **Units = 0:** digits from $\{1, \dots, 5\}$ and 5 *must* appear: place the 5 (3 ways), fill the two other spots from the remaining four digits: $4 \cdot 3 = 12 \rightarrow 36$. Total $48 + 36 = 84$. **Answer: D**

2022 AMC 8 #14 · gap method

Example 3. In how many ways can the letters in **BEEKEEPER** be rearranged so that two or more **E**s do not appear together?

- (A) 1 (B) 4 (C) 12 (D) 24 (E) 120



#6 Gap method

Key idea: "no two E's together" $\rightarrow$ arrange everyone else, then park the E's in the gaps. B, K, P, R arrange in $4! = 24$ ways. They create exactly 5 gaps ($_B_K_P_R_$), and we have exactly 5 identical E's — one per gap, 1 way. Total $24 \cdot 1 = 24$. **Answer: D**

Example 4. Professor Chang has nine different language books lined up on a bookshelf: two Arabic, three German, and four Spanish. How many ways are there to arrange the nine books on the shelf keeping the Arabic books together and keeping the Spanish books together?

- (A) 1440 (B) 2880 (C) 5760 (D) 182,440 (E) 362,880



#7 Bundling

#1 Multiplication rule

Key idea: two bundles. Tie the 2 Arabic books into one package ($2!$ inside) and the 4 Spanish books into another ($4! = 24$ inside). Now arrange 2 packages + 3 German books = 5 items: $5! = 120$. Total $2 \cdot 24 \cdot 120 = 5760$. **Answer: C**

Example 5. The "Middle School Eight" basketball conference has 8 teams.

Every season, each team plays every other conference team twice (home and away), and each team also plays 4 games against non-conference opponents. What is the total number of games in a season involving the "Middle School Eight" teams?

- (A) 60 (B) 88 (C) 96 (D) 144 (E) 160



#3 A or C?

#1 Multiplication rule

Key idea: a game is an *unordered* pair of teams. Conference games: $C_8^2 = 28$ pairs, each pair plays twice $\rightarrow 56$. Non-conference: 4 games $\times$ 8 teams = 32 (each such game involves only one conference team, so no double count). Total $56 + 32 = 88$. **Answer: B**

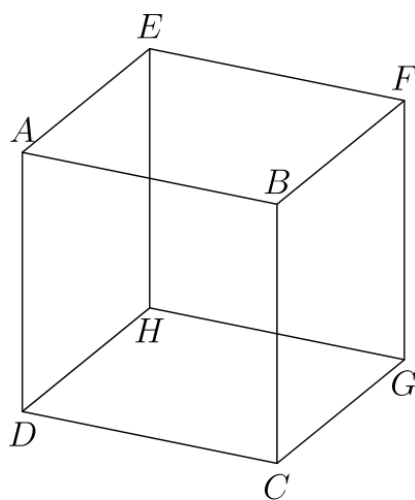
Practice

Try each one before opening the solution. Figures are from the official exams.

1. A girl has 5 shirts, 4 skirts, and 3 pairs of shoes. How many different outfits can she create?

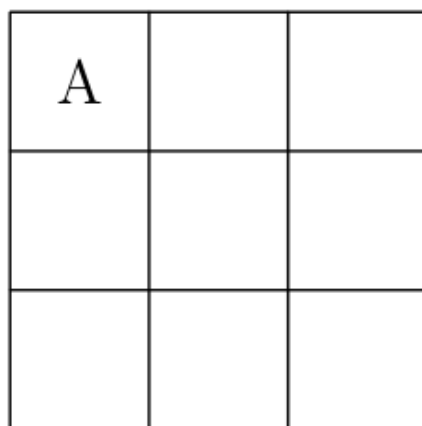
- (A) 120 (B) 60 (C) 15 (D) 12 (E) 20

2. How many pairs of parallel edges, such as $\overline{AB}$ and $\overline{GH}$ or $\overline{EH}$ and $\overline{FG}$, does a cube have?



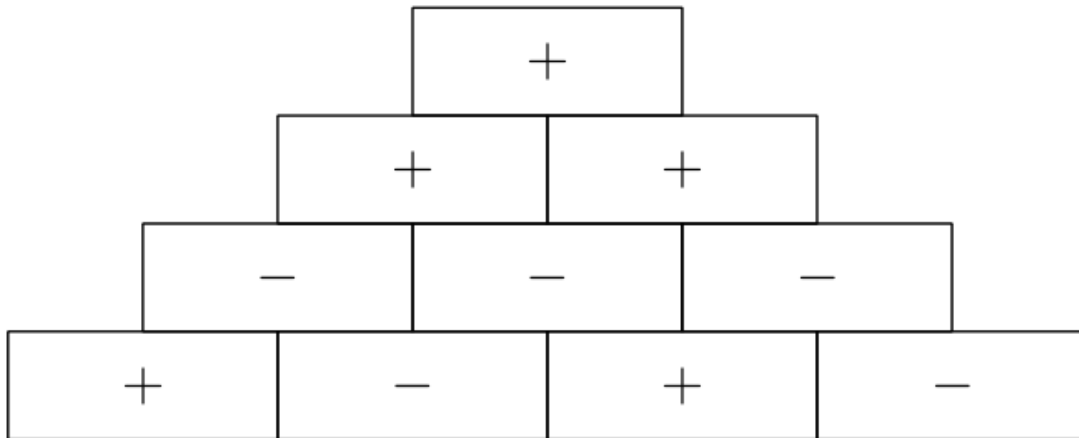
- (A) 6 (B) 12 (C) 18 (D) 24 (E) 36

3. Three A 's, three B 's, and three C 's are placed in the nine spaces so that each row and column contains one of each letter. If A is placed in the upper left corner, how many arrangements are possible?



- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

4. In a sign pyramid a cell gets a "+" if the two cells below it have the same sign, and it gets a "-" if the two cells below it have different signs. The diagram below illustrates a sign pyramid with four levels. How many possible ways are there to fill the four cells in the bottom row to produce a "+" at the top of the pyramid?



- (A) 2 (B) 4 (C) 8 (D) 12 (E) 16

5. How many 3-digit positive integers have digits whose product equals 24?

- (A) 12 (B) 15 (C) 18 (D) 21 (E) 24

6. Three girls and four boys are to be seated in a row containing seven chairs. If the chairs at both ends of the row must be occupied by girls, in how many different ways can the children be seated?

- (A) 120 (B) 720 (C) 5040 (D) 2520 (E) 240

7. Zara has a collection of 4 marbles: an Aggie, a Bumblebee, a Steelie, and a Tiger. She wants to display them in a row on a shelf, but does not want to put the Steelie and the Tiger next to one another. In how many ways can she do this?

- (A) 6 (B) 8 (C) 12 (D) 18 (E) 24

8. In how many ways can 5 books be arranged on a shelf if two of the books must remain together, but may be interchanged?

- (A) 12 (B) 24 (C) 48 (D) 96 (E) 5

9. How many 4-digit numbers greater than 1000 are there that use the four digits of 2012?

- (A) 6 (B) 7 (C) 8 (D) 9 (E) 12

10. In how many different ways can all the letters in INDIANA be arranged in a line? Assume that duplicate letters are indistinguishable.

- (A) 5040 (B) 2520 (C) 1260 (D) 630 (E) none of these

11. How many integers between 2020 and 2400 have four distinct digits arranged in increasing order? (For example, 2347 is one integer.)

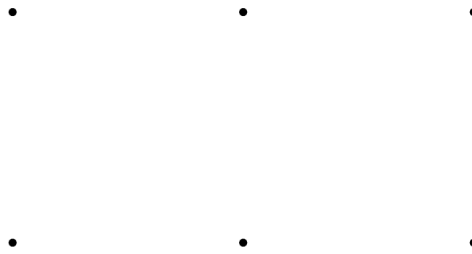
- (A) 9 (B) 10 (C) 15 (D) 21 (E) 28

12. Use four different colors to color the four rectangles A , B , C and D shown in the figure. No two rectangles sharing an edge may be the same color (colors may be reused otherwise). How many ways are there to color the rectangles?



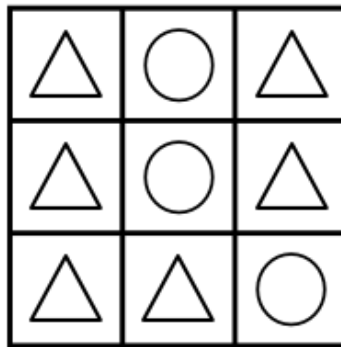
- (A) 120 (B) 100 (C) 84 (D) 64 (E) 24

13. How many distinct triangles can be drawn using three of the dots below as vertices?



- (A) 9 (B) 12 (C) 18 (D) 20 (E) 24

14. A $\triangle$ or $\bigcirc$ is placed in each of the nine squares in a 3-by-3 grid. Shown below is a sample configuration with three $\triangle$ s in a line. How many configurations will have three $\triangle$ s in a line and three $\bigcirc$ s in a line?

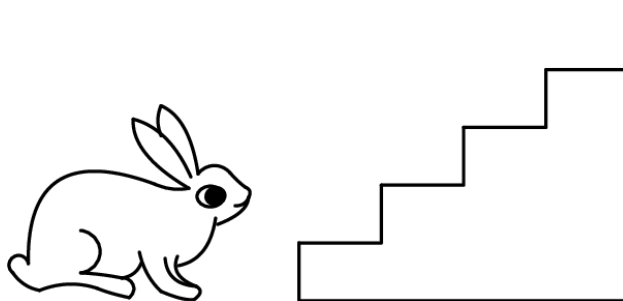


- (A) 39 (B) 42 (C) 78 (D) 84 (E) 96

NEW Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

15. Buzz Bunny is hopping up and down a set of stairs, one step at a time. In how many ways can Buzz start on the ground, make a sequence of 6 hops, and end up back on the ground? (For example, one sequence of hops is up-up-down-down-up-down.)

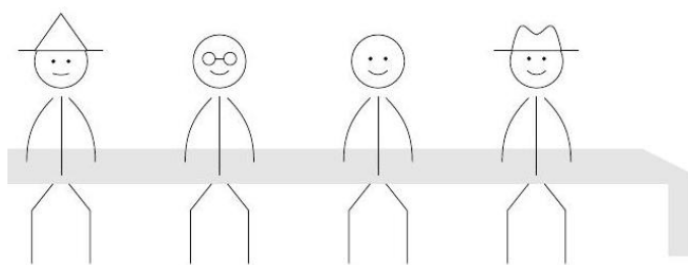


- (A) 4 (B) 5 (C) 6 (D) 8 (E) 12

16. The land of Catania uses gold coins and silver coins. Gold coins are 1 mm thick and silver coins are 3 mm thick. In how many ways can Taylor make a stack of coins that is 8 mm tall using any arrangement of gold and silver coins, assuming order matters?

- (A) 3 (B) 7 (C) 10 (D) 13 (E) 16

17. Four students are seated in a row. They chat with the people sitting next to them, then rearrange themselves so that they are no longer seated next to any of the same people. How many rearrangements are possible?



- (A) 2 (B) 4 (C) 9 (D) 12 (E) 24

★ **Challenge (AMC 10 level)**

2018 AMC 10A #4 · gap method

C1. How many ways can a student schedule 3 mathematics courses — algebra, geometry, and number theory — in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

(A) 3 (B) 6 (C) 12 (D) 18 (E) 24

2017 AMC 10A #19 · gap method

C2. Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. All five must sit in a row of 5 chairs. In how many ways can they be seated?

(A) 12 (B) 16 (C) 28 (D) 32 (E) 40

Sources: prep-course Lecture 12 (video + board notes + 38 worked examples, translated to English) and official AMC 8 problems from the local bank (kb/content/banks/). Answers verified against official AoPS answer keys where available.

Answer key

Example	1	2	3	4	5													
	D	D	D	C	B													
Practice	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
	B	C	C	C	D	B	C	C	D	D	C	C	C	D	B	D	A	
Challenge	C1	C2																
	E	C																

Solutions to practice problems

Practice 1.



#1 Multiplication rule

Three required steps: shirt, skirt, shoes $\rightarrow 5 \cdot 4 \cdot 3 = 60$. **Answer: B**

Practice 2.



#3 A or C?

#2 Addition rule

The 12 edges split into 3 directions of 4 parallel edges each. Within one direction: $C_4^2 = 6$ pairs. Three directions $\rightarrow 3 \cdot 6 = 18$. (Or: 12 edges $\times$ 3 parallel partners $\div 2$ for double counting = 18.) **Answer: C**

Practice 3.



#1 Multiplication rule

Step 1: place the other two A 's — they must avoid row 1 and column 1 *and* each other's row/column: only 2 ways (the two diagonals of the lower-right 2×2 block). Step 2: fill column 1 below the A : BC or CB — 2 ways. Everything else is then forced. $2 \cdot 2 = 4$. **Answer: C**

Practice 4.



#1 Multiplication rule

#5 Complement

Work top-down: the top "+" forces row 2 to be equal signs — pick its left cell (2 ways), the right is forced. In each lower row, once the row above is known, the leftmost cell is free (2 ways) and the rest are forced. So $2 \cdot 2 \cdot 2 = 8$. *Slicker*: all $2^4 = 16$ bottom rows are possible and by symmetry half give "+": $16 \div 2 = 8$. **Answer: C**

Practice 5.



#2 Addition rule

#8 Repeated items

Digit triples with product 24: $\{2, 2, 6\}$, $\{2, 3, 4\}$, $\{1, 3, 8\}$, $\{1, 4, 6\}$. $\{2, 2, 6\}$ has a repeat: $3!/2! = 3$ numbers. Each all-distinct triple gives $3! = 6$: three of them $\rightarrow 18$. Total $3 + 18 = 21$. **Answer: D**

Practice 6.



#3 A or C?

#1 Multiplication rule

Fill the special seats first: pick two of the three girls *in order* for the two ends: $A_3^2 = 6$. The remaining 5 children fill the middle: $5! = 120$. Total $6 \cdot 120 = 720$. **Answer: B**

Practice 7.



#6 Gap method

Arrange the two friendly marbles: $2! = 2$; they make 3 gaps; drop Steelie and Tiger into different gaps in order: $A_3^2 = 6$. Total $2 \cdot 6 = 12$. (Complement check: $4! - 2 \cdot 3! = 24 - 12 = 12$ ✓) **Answer: C**

Practice 8.



#7 Bundling

Bundle the two books ($2!$ inside), then arrange the bundle with the other three: $4! = 24$. Total $2 \cdot 24 = 48$. **Answer: C**

Practice 9.



#8 Repeated items

#5 Complement

Digits 2, 0, 1, 2 (two 2's): total arrangements $\frac{4!}{2!} = 12$; those starting with 0: arrange 2, 1, 2 $\rightarrow \frac{3!}{2!} = 3$.
 $12 - 3 = 9$. **Answer: D**

Practice 10.



#8 Repeated items

Seven letters with two I's, two N's, two A's: $\frac{7!}{2! 2! 2!} = \frac{5040}{8} = 630$. **Answer: D**

Practice 11.



#3 A or C?

Increasing forces the first two digits to be 2, 3. The last two digits are any two of $\{4, \dots, 9\}$ — and once chosen, their order is *forced* (increasing), so it's a combination: $C_6^2 = 15$. **Answer: C**

Practice 12.



#1 Multiplication rule

#2 Addition rule

Only diagonal pairs (A, D and B, C) may share a color. Color in the order A, B, C, D and split on whether C copies B : $C = B$: A : 4, B : 3 ($\neq A$), C : 1, D : 3 ($\neq B=C$) $\rightarrow 4 \cdot 3 \cdot 1 \cdot 3 = 36$. $C \neq B$: $C \neq A, B \rightarrow 2$; $D \neq B, C \rightarrow 2 \rightarrow 4 \cdot 3 \cdot 2 \cdot 2 = 48$. Total $36 + 48 = 84$. (Instructor's coloring order: start from the region touching the most others.) **Answer: C**

Practice 13.



#5 Complement

#3 A or C?

Any 3 of the 6 dots: $C_6^3 = 20$. Subtract the degenerate "triangles": 3 collinear dots — only the top row and the bottom row, 2 cases. $20 - 2 = 18$. **Answer: C**

Practice 14.



#2 Addition rule

#5 Complement

A diagonal line of $\triangle$ s would block any line of $\bigcirc$ s, so both lines are rows or both are columns — and by symmetry count rows, then double. **Two rows of $\triangle$** : choose them, $C_3^2 = 3$; the last row must be all $\bigcirc$: 3 ways total. **One row of $\triangle$** : $C_3^1 = 3$; the other two rows: either both all- $\bigcirc$ (1 way), or one all- $\bigcirc$ (2 choices) and the last row mixed — not all- $\triangle$, not all- $\bigcirc$: $2^3 - 2 = 6$ ways. That's $3(1 + 2 \cdot 6) = 39$. Rows total $3 + 39 = 42$; doubling for columns: 84. **Answer: D**

Practice 15.



#2 Addition rule

Six hops ending on the ground means 3 ups and 3 downs — but Buzz can never dip *below* the ground, so list carefully (U = up, D = down): UUUDDD, UUDUDD, UUDUD, UDUUDD, UDUDUD — the other arrangements of 3 U's and 3 D's all sink below ground at some point. That's 5. **Answer: B**

Practice 16.



#2 Addition rule

#8 Repeated items

Case on the number of silver coins s (each 3 mm), with g gold coins: $g + 3s = 8$. $s = 0$: 8 golds $\rightarrow$ 1 stack. $s = 1$: 5 golds + 1 silver $\rightarrow \frac{6!}{5!} = 6$ stacks. $s = 2$: 2 golds + 2 silvers $\rightarrow \frac{4!}{2!2!} = 6$ stacks. Total $1 + 6 + 6 = 13$. **Answer: D**

Practice 17.



#2 Addition rule

#5 Complement

Call the original row A, B, C, D ; the forbidden neighbor pairs are AB, BC, CD . B and C each had two neighbors, so each can only sit next to one other person: B next only to D , C next only to A — so the new row must glue BD and AC together, with B, C on the *outside* (they can't touch each other or their old partners in the middle): $B D A C$ and $C A D B$. Just 2. **Answer: A**

Practice C1.



#6 Gap method

#1 Multiplication rule

The three non-math periods create 4 gaps ($_x_x_x_$); choose 3 gaps for the math courses: $C_4^3 = 4$. Then assign the three *different* courses to those periods: $3! = 6$. Total $4 \cdot 6 = 24$. **Answer: E**

Practice C2.



#6 Gap method

#2 Addition rule

Seat A, B, C first, then insert D, E into the gaps so that every forbidden touch is broken. Case on where A sits relative to B, C : **A on a side** (orders ABC -type, B, C swappable: 2 ways each side): the A - B gap *must* get D or E (2 ways); the remaining person has 3 slots. $2 \cdot 2 \cdot 3 = 12$ per side $\rightarrow 24$ for both sides. **A in the middle** (BAC , 2 ways): both gaps around A must be filled by D and E : 2 ways. $2 \cdot 2 = 4$. Total $24 + 4 = 28$. **Answer: C**