Angles & Triangles · Lecture 01 · `geo-angles-triangles`

📺 **Course materials — Lecture 01:** lecture video · handout PDF (Lecture 1) · board notes (01.jpg) · homework answer key (Chapter 1) — all in the `amc8/video` folder. · 🖨️ **Printable PDF** (answers & solutions at the back)

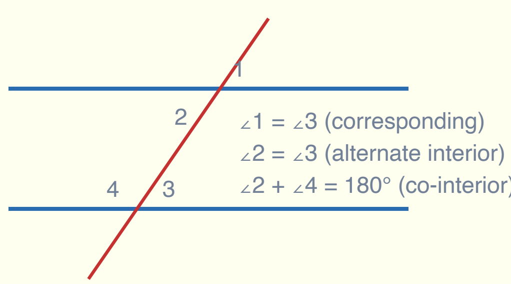
🔑 Key facts

The three angles of a triangle sum to $180°$; an **exterior angle equals the sum of the two remote interior angles** — that one identity solves half the problems in this lecture. An n-gon's interior angles sum to $(n-2)\cdot 180°$ (split it into triangles from one vertex), and the exterior angles of *any* polygon sum to $360°$. Finally, sides are constrained too: in every triangle the sum of any two sides is greater than the third (the **triangle inequality**).

🧰 Instructor's toolbox (from the lecture video & board notes)

1 · Angle vocabulary → equations. Acute $< 90°$, right $= 90°$, obtuse, straight $= 180°$, and the **reflex** angle ($> 180°$) on the far side of any angle. Two angles are **supplementary** if they sum to $180°$ and **complementary** if they sum to $90°$. The instructor's move: name the unknown x and translate the words directly — "twice the supplement is 27 more than five times the complement" becomes $2(180-x) = 5(90-x) + 27$ (Practice 10).

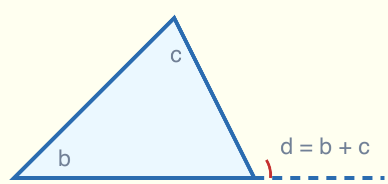
2 · Parallel lines + a transversal: three angle pairs. Corresponding angles are equal, alternate interior angles are equal, and **co-interior angles (same side) sum to** $180°$. That last one is the workhorse: in Practice 6, $\angle 1$ and $\angle 5$ are co-interior, so $(7x+10)+3x = 180$ and $x = 17$ — one equation, done.



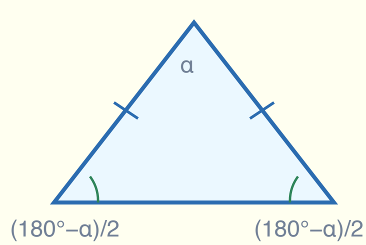
3 · No parallel line? Draw one. When a point sits *between* two parallel rays (a zig-zag / "Z with a bump"), draw the line through that point parallel to both. The bump angle splits into two pieces, each handled by tip #2. That is the entire solution of Practice 8 — and the same auxiliary line proves the 180° angle sum itself.

4 · Angle sums: triangle 180°, n-gon $(n-2)\cdot 180^\circ$. The proof is tip #3: a parallel line through the apex turns the three angles into a straight line. For any polygon, split it into triangles from one vertex: quadrilateral $= 2 \times 180^\circ = 360^\circ$ (even a non-convex "arrow" — Practice 2), pentagon $= 540^\circ$, hexagon $= 720^\circ$. A full turn around any interior point is 360° — that closes out Practice 3 and Practice 12.

5 · Exterior angle = sum of the two remote interior angles. The instructor calls it the fast lane: it replaces *two* unknown angles with *one* known angle, no algebra. Example 2 is instant ($\angle ADB = 70^\circ + 70^\circ$), Practice 5 chains it twice through a star, and Practice 3 uses it to collapse six scattered angles into one full turn. Exterior angles of any polygon sum to 360°.

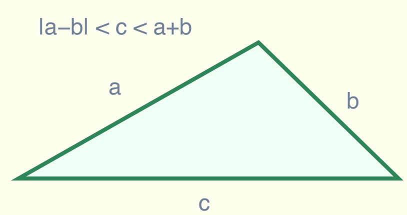


6 · Isosceles: equal sides ⇄ equal angles. The board's slogan: "equal sides face equal angles", and it runs both ways. Given the apex angle α, each base angle is $\dfrac{180^\circ - \alpha}{2}$ (Example 3: apex 36° → base angles 72°; Example 4: apex 20° → base angles 80°). Spot the equal sides *first*, mark the equal angles, then chase.



7 · Equilateral: all sides equal, all angles 60°. Two ways it hides on exams: an isosceles triangle with any 60° angle is automatically equilateral (Practice 1: $DA = CD$ and $\angle ADC = 60^\circ$ force $AC = DA$); and chains of equilateral triangles share side lengths, so perimeters can be read off by counting (Practice 4). Regular-polygon angles to know cold: 60° (triangle), 90°, 108°, 120° (hexagon).

8 · Triangle inequality — sides have rules too. Any two sides sum to more than the third. Given two sides a, b, the third side lives in the open range $|a - b| < c < a + b$ (Example 1: sides 5 and 19 → $14 < c < 24$ — the instructor derives it with the two cases "which side is longest"). Add a constant to every part of the inequality to get a *perimeter* range. Checking whether three given lengths form a triangle? Only test the two smallest against the largest.



9 · Angle chasing with algebra: hunt the SUM, not each angle. Often the individual angles cannot be found — and don't need to be. In Example 5 the given difference $\angle CAB - \angle ABC = 40^\circ$ simplifies to $2\angle BAD = 40^\circ$; in Practice 9 you only ever need $\angle RXY + \angle RYX$, which the bisectors tie to $\frac{1}{2}(180^\circ - a)$. Write what you want in terms of sums the problem controls, and let the extra unknowns cancel.

📖 Worked examples

2015 AMC 8 #8 · triangle inequality

Example 1. What is the smallest whole number larger than the perimeter of any triangle with a side of length 5 and a side of length 19?

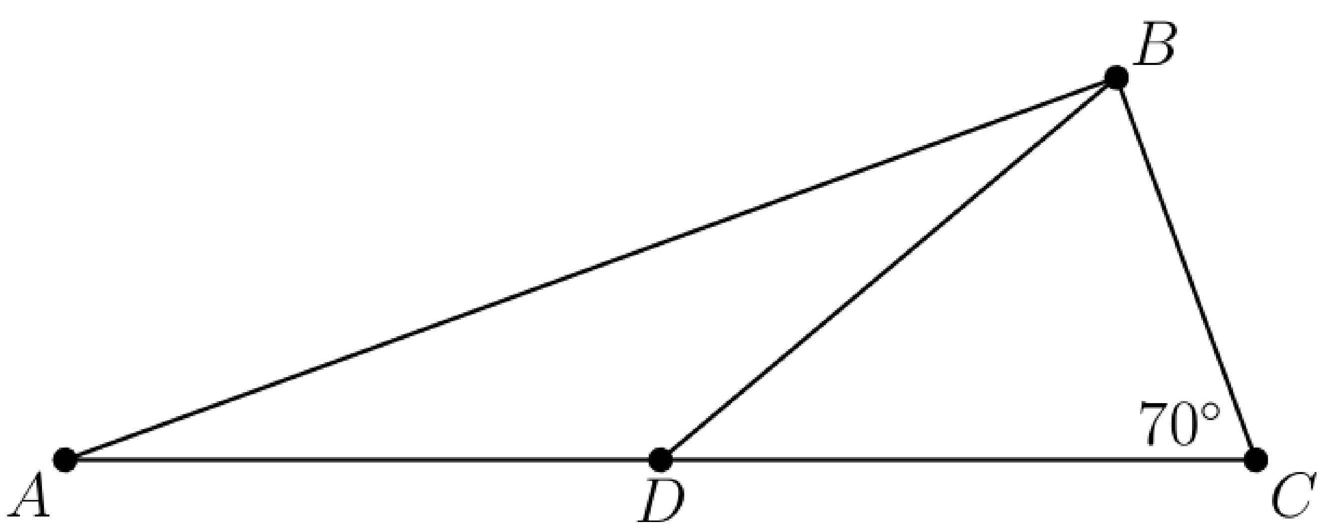
(A) 24 **(B)** 29 **(C)** 43 **(D)** 48 **(E)** 57

🧰 #8 Triangle inequality

Key idea: pin down the third side, then shift the whole inequality. Third side c: $19-5<c<19+5$, so $14<c<24$. Add $5+19=24$ to every part: $38<P<48$. The perimeter can get as close to 48 as it likes but never reaches it, so the smallest whole number larger than every possible perimeter is 48. **Answer: D**

2014 AMC 8 #9 · exterior angle

Example 2. In $\triangle ABC$, D is a point on side $\overline{AC}$ such that $BD=DC$ and $\angle BCD$ measures 70°. What is the degree measure of $\angle ADB$?



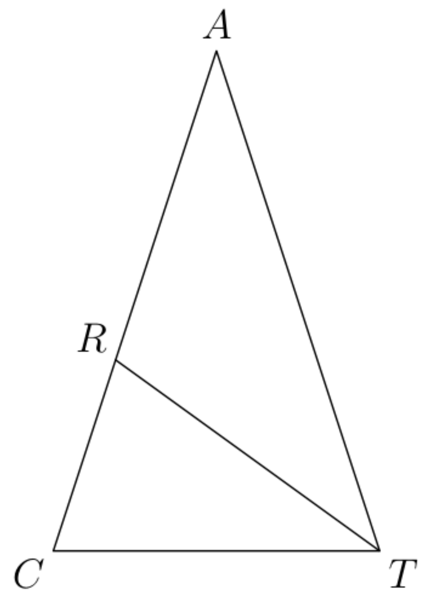
(A) 100 **(B)** 120 **(C)** 135 **(D)** 140 **(E)** 150

🧰 #6 Isosceles #5 Exterior angle

Key idea: $BD=DC$ makes $\triangle BDC$ isosceles, so $\angle DBC=\angle DCB=70^\circ$. Now $\angle ADB$ is the exterior angle of $\triangle BDC$ at D: $\angle ADB=\angle DBC+\angle DCB=70^\circ+70^\circ=140^\circ$ — no other angle in the figure is ever computed. **Answer: D**

2000 AMC 8 #13 · isosceles + bisector

Example 3. In triangle CAT, we have $\angle ACT = \angle ATC$ and $\angle CAT = 36^\circ$. If $\overline{TR}$ bisects $\angle ATC$, then $\angle CRT =$?



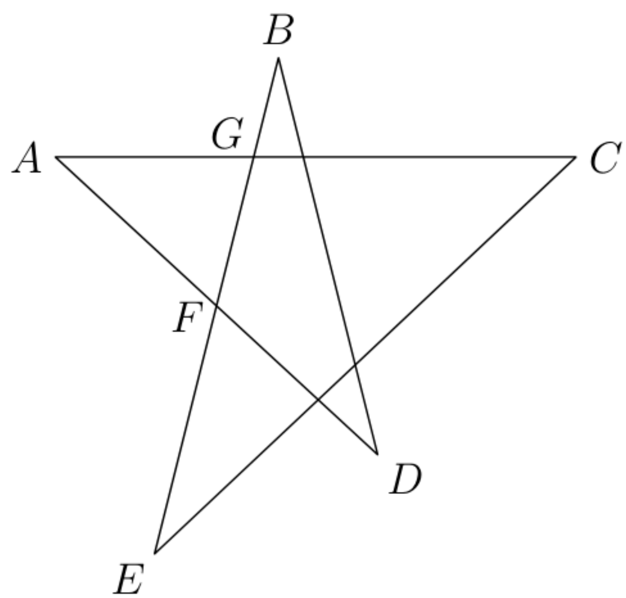
(A) 36° **(B)** 54° **(C)** 72° **(D)** 90° **(E)** 108°

🧰 #6 Isosceles #5 Exterior angle

Key idea: apex 36° → base angles $\angle ACT = \angle ATC = \dfrac{180^\circ - 36^\circ}{2} = 72^\circ$. The bisector gives $\angle ATR = 36^\circ$. Then $\angle CRT$ is the exterior angle of $\triangle ART$ at R: $\angle CRT = \angle A + \angle ATR = 36^\circ + 36^\circ = 72^\circ$. (Or inside $\triangle CRT$: $180^\circ - 72^\circ - 36^\circ = 72^\circ$.) **Answer: C**

2000 AMC 8 #24 · exterior angle

Example 4. If $\angle A = 20^\circ$ and $\angle AFG = \angle AGF$, then $\angle B + \angle D = ?$



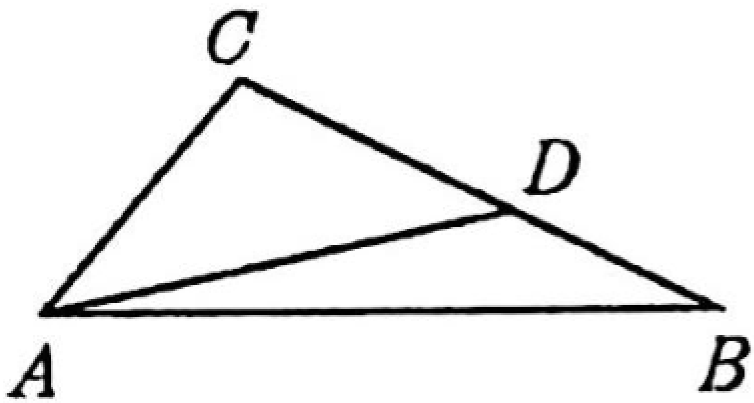
(A) 48° **(B)** 60° **(C)** 72° **(D)** 80° **(E)** 90°

#6 Isosceles #5 Exterior angle

Key idea: $\triangle AFG$ is isosceles with apex 20°, so $\angle AFG = \angle AGF = \dfrac{180^\circ - 20^\circ}{2} = 80^\circ$. But $\angle AFG$ is also an exterior angle of $\triangle BFD$ (the small triangle at F on line BE), so $\angle B + \angle D = \angle AFG = 80^\circ$ — the sum pops out even though $\angle B$ and $\angle D$ are individually unknown. **Answer: D**

course mock · angle chasing

Example 5. In $\triangle ABC$, point D lies on $\overline{CB}$ with $AC = CD$, and $\angle CAB - \angle ABC = 40^\circ$. What is $\angle BAD$?



(A) 15° **(B)** 20° **(C)** 30° **(D)** 35° **(E)** 40°

🧰 #9 Hunt the sum #6 Isosceles #5 Exterior angle

Key idea: rewrite the given difference until everything except $\angle BAD$ cancels. $AC = CD$ gives $\angle CAD = \angle CDA$. Split: $\angle CAB = \angle CAD + \angle DAB$, so

$$\angle CAD + \angle DAB - \angle ABC = 40^\circ.$$

Replace $\angle CAD$ by $\angle CDA$, and $\angle CDA$ is the exterior angle of $\triangle ABD$ at D: $\angle CDA = \angle DAB + \angle ABC$. Substituting,

$$(\angle DAB + \angle ABC) + \angle DAB - \angle ABC = 2\angle DAB = 40^\circ,$$

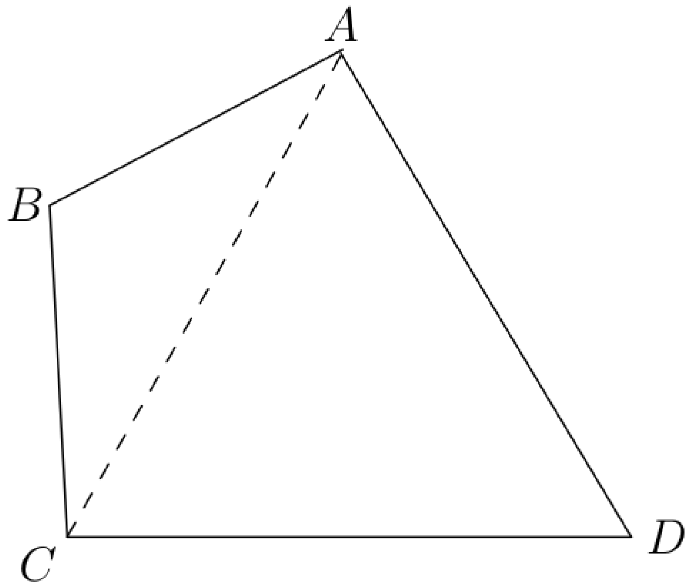
so $\angle BAD = 20^\circ$ — neither $\angle CAB$ nor $\angle ABC$ is ever known alone. **Answer: B**

✏️ Practice

Try each one before opening the solution. Figures are from the official exams and course handouts.

2005 AMC 8 #9

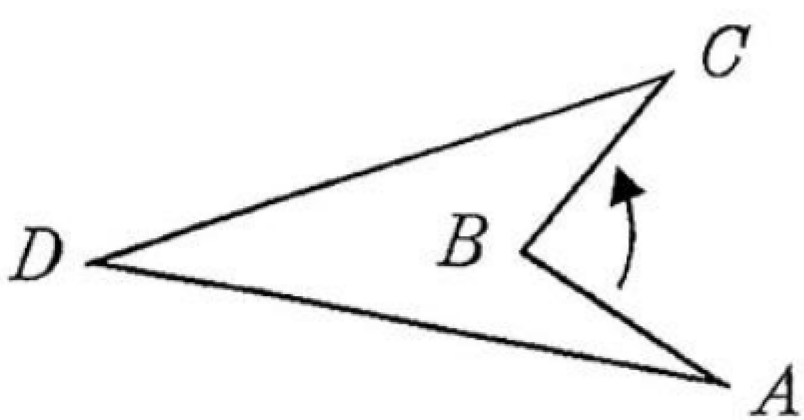
1. In quadrilateral $ABCD$, sides $\overline{AB}$ and $\overline{BC}$ both have length 10, sides $\overline{CD}$ and $\overline{DA}$ both have length 17, and the measure of angle ADC is 60°. What is the length of diagonal $\overline{AC}$?



(A) 13.5 **(B)** 14 **(C)** 15.5 **(D)** 17 **(E)** 18.5

course mock · non-convex quadrilateral

2. The tip of an arrow has the shape shown below. If the angle ABC marked by the curved arrow is an acute angle, then the sum of the interior angles of quadrilateral $ABCD$:

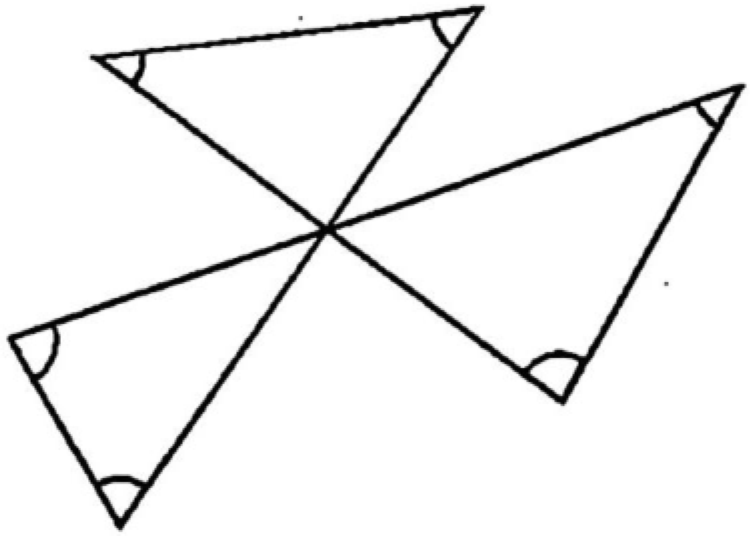


(A) is less than 180° **(B)** is less than 360° **(C)** is exactly 360° **(D)** is more than 360°

(E) none of the above

course mock · pinwheel

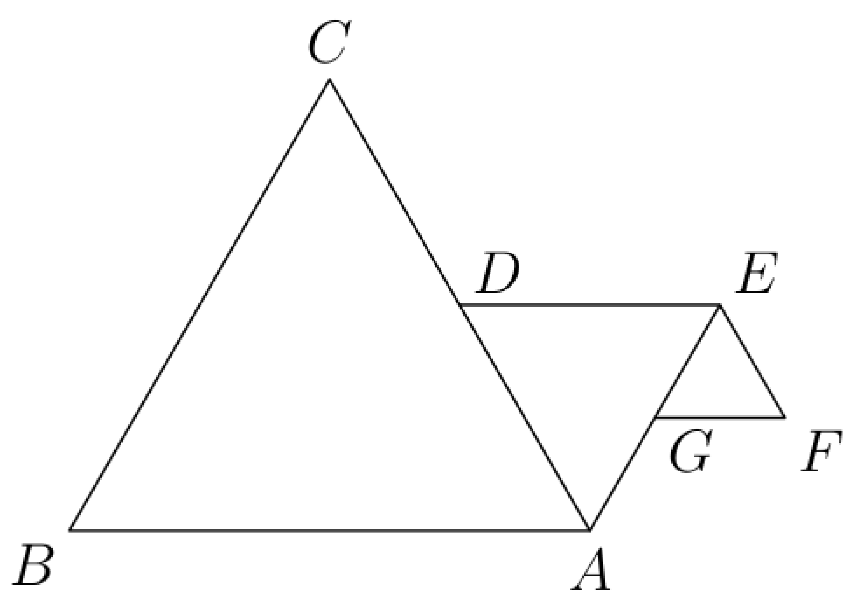
3. In the figure below, the sum of all the marked angles is:



(A) 180° **(B)** 360° **(C)** 540° **(D)** 270° **(E)** cannot be determined

2000 AMC 8 #15

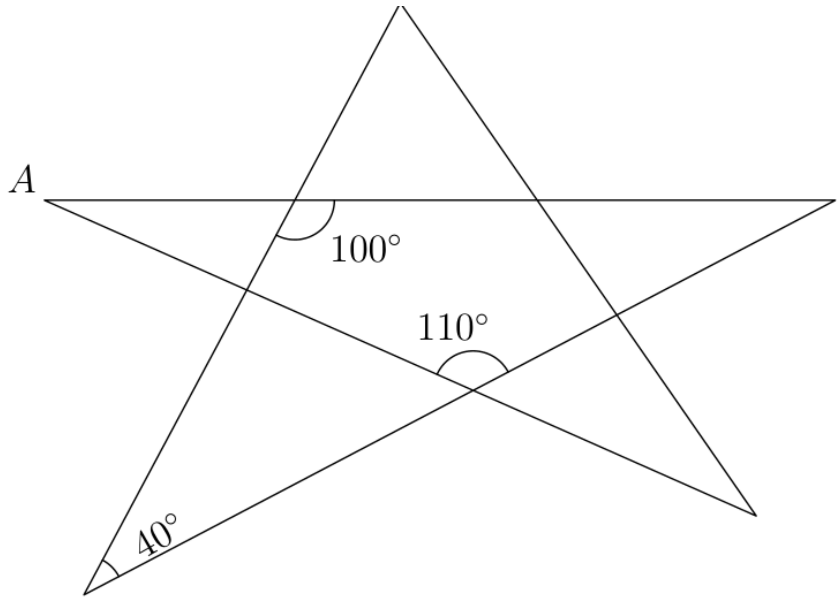
4. Triangles ABC, ADE, and EFG are all equilateral. Points D and G are midpoints of $\overline{AC}$ and $\overline{AE}$, respectively. If $AB = 4$, what is the perimeter of figure $ABCDEFG$?



(A) 12 **(B)** 13 **(C)** 15 **(D)** 18 **(E)** 21

1999 AMC 8 #21

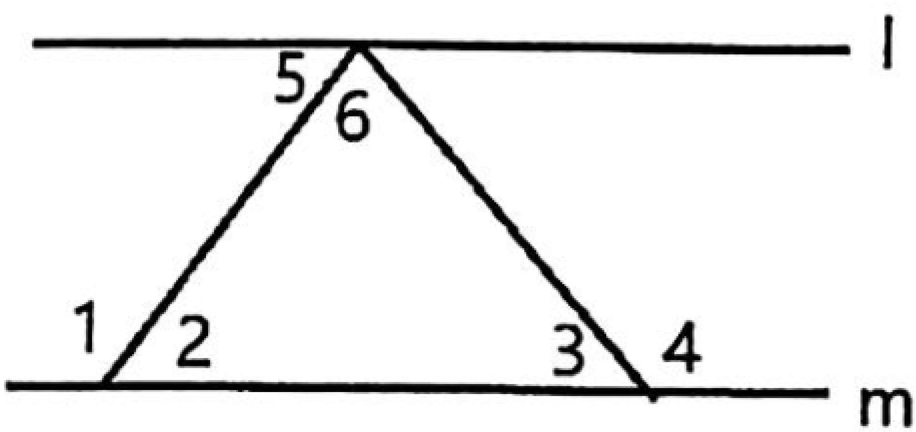
5. The degree measure of angle A is:



(A) 20 **(B)** 30 **(C)** 35 **(D)** 40 **(E)** 45

course mock · parallel lines

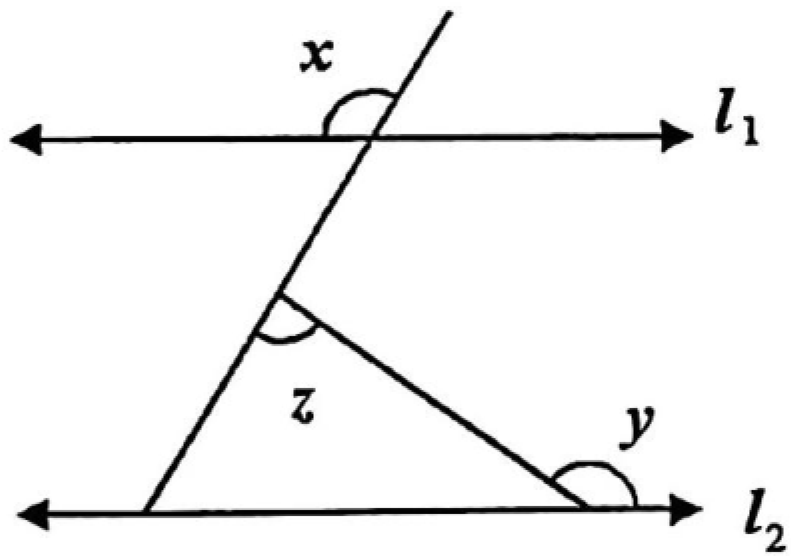
6. In the figure, $\angle 1 = 7x + 10$, $\angle 5 = 3x$, and $l \parallel m$. The measure of $\angle 2$ equals:



(A) 17° **(B)** 51° **(C)** 87° **(D)** 129° **(E)** 139°

course mock · parallel lines

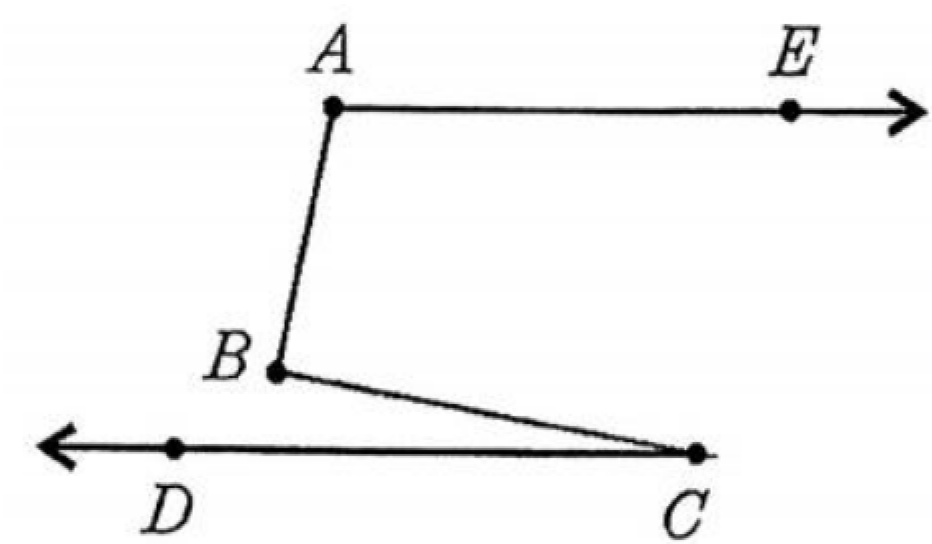
7. x, y, and z are the measures of the angles shown, and $l_1 \parallel l_2$. The measure of x is:



(A) $180^\circ - x$ **(B)** $180^\circ - z$ **(C)** $180^\circ - z + y$ **(D)** $180^\circ + z - y$ **(E)** $z + y - 180^\circ$

course mock · auxiliary parallel line

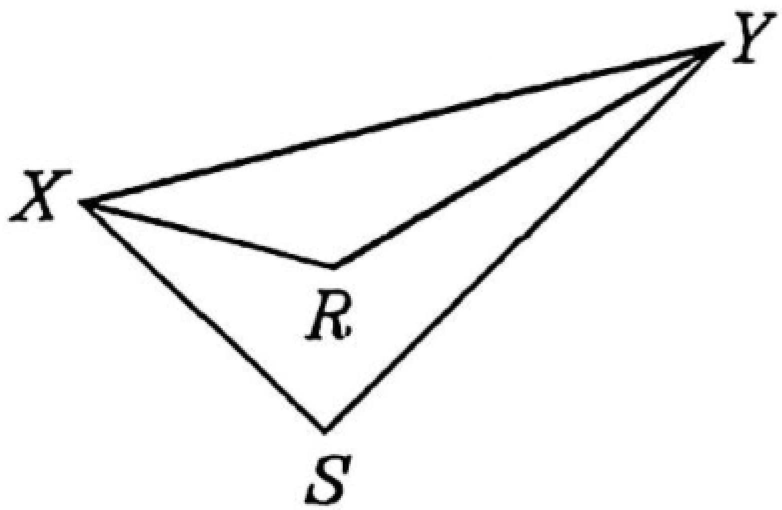
8. In the figure shown, $\overrightarrow{AE}$ is parallel to $\overrightarrow{CD}$ and B is a point between the rays. If $\angle BAE = 100^\circ$ and $\angle ABC = 90^\circ$, find the measure of $\angle BCD$.



(A) 5° **(B)** 90° **(C)** 100° **(D)** 30° **(E)** 10°

course mock · angle bisectors

9. In the figure, $\overline{XR}$ bisects $\angle YXS$, $\overline{YR}$ bisects $\angle XYS$, and $\angle S = a$. Express the measure of $\angle R$ in terms of a.



(A) $90 + \frac{a}{2}$ **(B)** $60 - \frac{a}{3}$ **(C)** $30 + \frac{2a}{3}$ **(D)** $180 + \frac{a}{2}$ **(E)** $-30 + \frac{a}{3}$

course mock · supplement & complement

10. What is the measure of an acute angle if twice the measure of its supplement is 27 more than five times the measure of its complement?

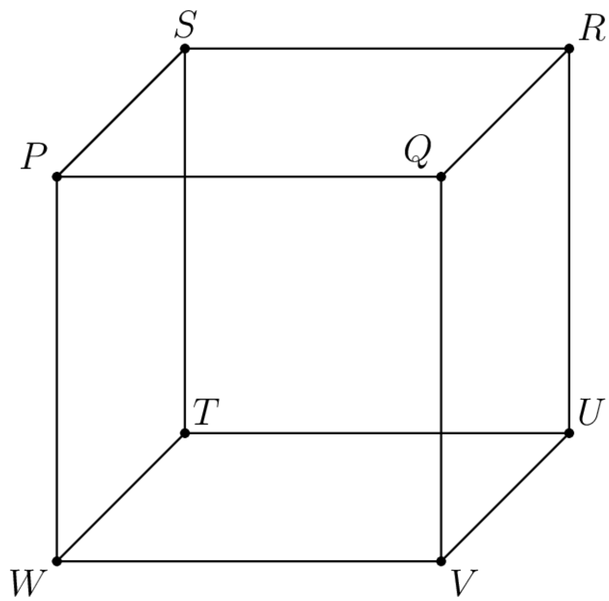
(A) 17° **(B)** 23° **(C)** 31° **(D)** 39° **(E)** 47°

🆕 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2024 AMC 8 #20

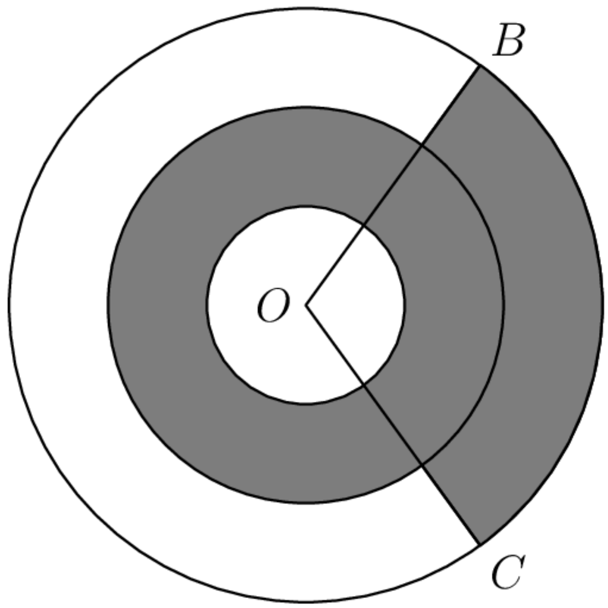
11. Any three vertices of the cube $PQRSTUVW$, shown in the figure below, can be connected to form a triangle. (For example, vertices P, Q, and R can be connected to form $\triangle PQR$.) How many of these triangles are equilateral and contain P as a vertex?



(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 6

2024 AMC 8 #18

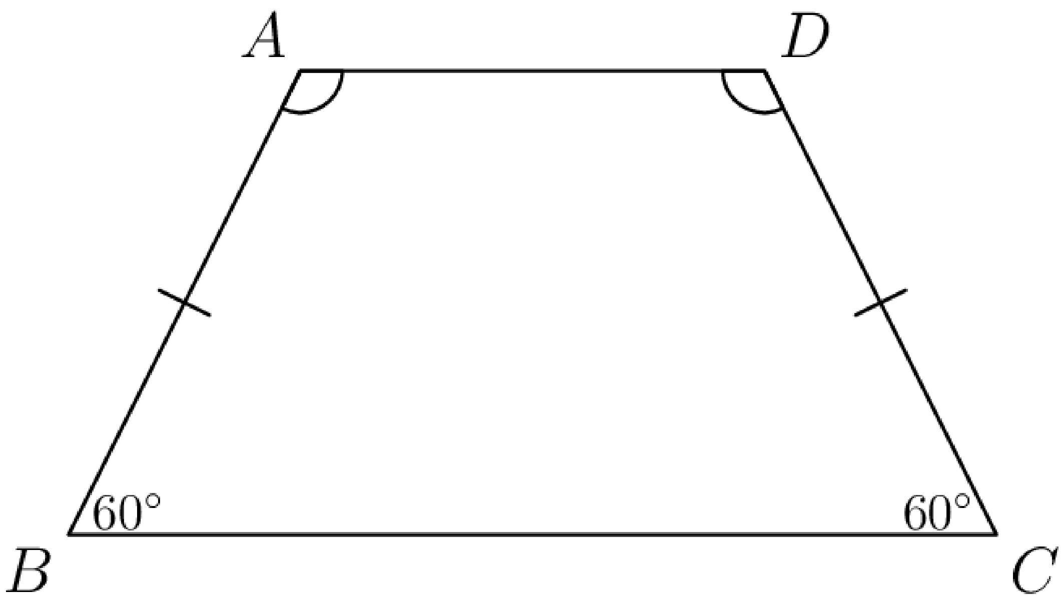
12. Three concentric circles centered at O have radii of 1, 2, and 3. Points B and C lie on the largest circle. The region between the two smaller circles is shaded, as is the portion of the region between the two larger circles bounded by central angle BOC, as shown in the figure below. Suppose the shaded and unshaded regions are equal in area. What is the measure of $\angle BOC$ in degrees?



(A) 108 **(B)** 120 **(C)** 135 **(D)** 144 **(E)** 150

2025 AMC 8 #24

13. In trapezoid $ABCD$, angles B and C measure 60° and $AB = DC$. The side lengths are all positive integers, and the perimeter of $ABCD$ is 30 units. How many non-congruent trapezoids satisfy all of these conditions?



(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Sources: prep-course Lecture 01 (video + board notes + 15 worked examples, translated to English) and official AMC 8 problems from the local bank (`kb/content/banks/`). Answers verified against official AoPS answer keys where available; course-mock answers follow the course solution manual.

🗝️ Answer key

Example	1	2	3	4	5								
	D	D	C	D	B								
Practice	1	2	3	4	5	6	7	8	9	10	11	12	13
	D	C	B	C	B	B	D	E	A	D	D	A	E

📝 Solutions to practice problems

Practice 1.

🧰 #7 Equilateral #6 Isosceles

Look only at $\triangle ACD$: $DA = DC = 17$ with apex $\angle ADC = 60°$, so the base angles are $\frac{180°-60°}{2} = 60°$ each — all three angles are $60°$, the triangle is equilateral, and $AC = 17$. The sides of length 10 are a decoy. **Answer: D**

Practice 2.

🧰 #4 Angle sums

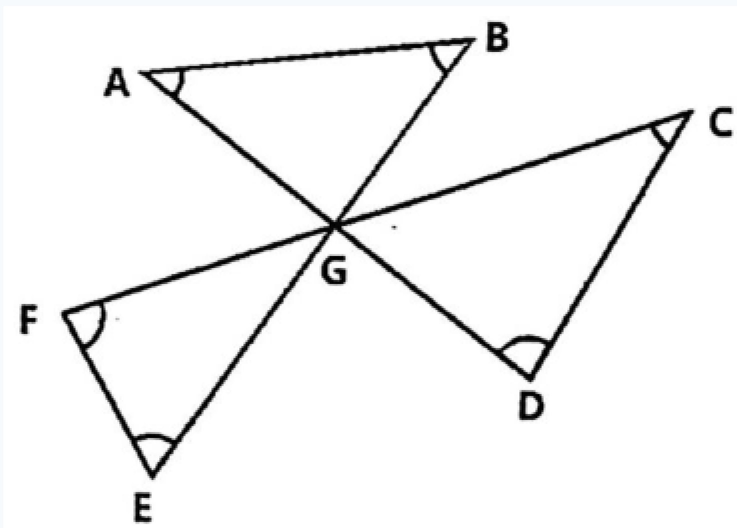
Connect $\overline{DB}$: the quadrilateral splits into $\triangle ABD$ and $\triangle CBD$, so its interior angles sum to $2 \times 180° = 360°$ — exactly as for any quadrilateral. The catch the arrow is testing: at the notch B the *interior* angle is the reflex angle ($> 180°$), not the acute angle the arrow marks. Count the true interior angles and the sum is exactly $360°$. **Answer: C**

Practice 3.

🧰 #5 Exterior angle #4 Angle sums

Label the triangles $\triangle ABG$, $\triangle CDG$, $\triangle EFG$ meeting at the center G (figure below). Each marked pair is swallowed by an exterior angle at the center: $\angle A + \angle B = \angle BGD$, $\angle C + \angle D = \angle DGF$, $\angle E + \angle F = \angle FGB$. The three center angles make a full turn:

$$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F = \angle BGD + \angle DGF + \angle FGB = 360°.$$



Answer: B

Practice 4.

🧰 #7 Equilateral

Equilateral triangles share all their side lengths: $\triangle ABC$ has side 4, $\triangle ADE$ has side $AD = \frac{1}{2}AC = 2$, $\triangle EFG$ has side $EG = \frac{1}{2}AE = 1$. Walk the boundary: $AB + BC + CD + DE + EF + FG + GA = 4 + 4 + 2 + 2 + 1 + 1 + 1 = 15$. **Answer: C**

Practice 5.

🧰 #5 Exterior angle

Chain the exterior angle theorem twice through the star. In the bottom-left triangle, the $110°$ angle is exterior: $110° = 40° + x$, so the third marked crossing angle is $x = 70°$. In the triangle containing A, the $100°$ angle is exterior: $100° = 70° + \angle A$, so $\angle A = 30°$. **Answer: B**

Practice 6.

🧰 #2 Parallel lines

$\angle 1$ and $\angle 5$ are co-interior angles between the parallels, so they sum to $180°$: $(7x + 10) + 3x = 180$, giving $x = 17$ and $\angle 1 = 129°$. Then $\angle 2 = 180° - \angle 1 = 51°$. **Answer: B**

Practice 7.

🧰 #2 Parallel lines #5 Exterior angle

The angle the transversal makes below l_1 (alternate side) is $180° - x$, and it sits inside the small triangle on l_2 whose exterior angle is y:

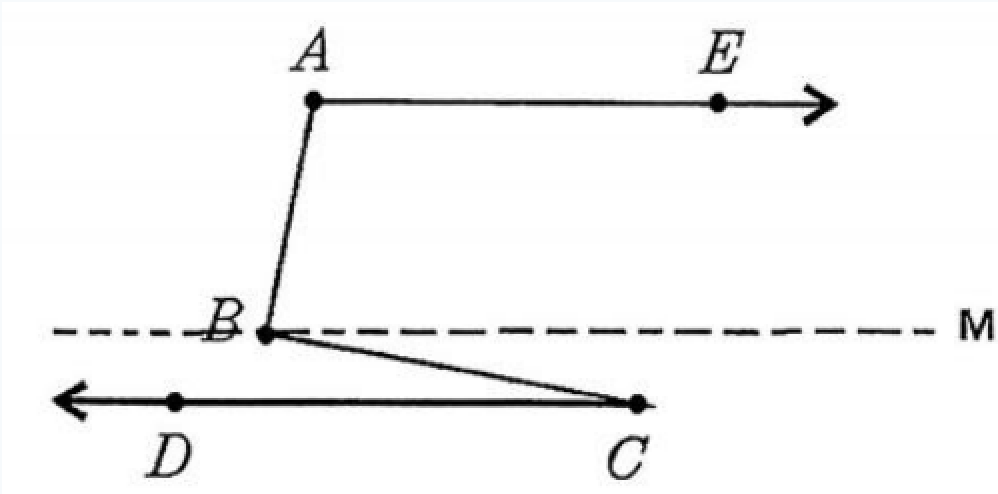
$$y = z + (180° - x) \implies x = 180° + z - y.$$

Answer: D

Practice 8.

🧰 #3 Draw a parallel | #2 Parallel lines

Draw line M through B parallel to both rays (dashed below). Above it: $\angle ABM$ and $\angle BAE$ are co-interior, so $\angle ABM = 180^\circ - 100^\circ = 80^\circ$. That leaves $\angle MBC = 90^\circ - 80^\circ = 10^\circ$, and by alternate angles $\angle BCD = \angle MBC = 10^\circ$.



Answer: E

Practice 9.

🧰 #9 Hunt the sum | #4 Angle sums

Never find the four half-angles separately — only their sum matters.

$$\angle RXY + \angle RYX = \tfrac{1}{2}(\angle SXY + \angle SYX) = \tfrac{1}{2}(180^\circ - a) = 90^\circ - \tfrac{a}{2}.$$

Then in $\triangle XYR$: $\angle R = 180^\circ - \left(90^\circ - \frac{a}{2}\right) = 90^\circ + \frac{a}{2}$. **Answer: A**

Practice 10.

🧰 #1 Vocabulary → equations

Translate word for word with the angle as x: supplement $= 180 - x$, complement $= 90 - x$:

$$2(180 - x) = 5(90 - x) + 27 \implies 360 - 2x = 477 - 5x \implies 3x = 117,$$

so $x = 39^\circ$ (an acute angle, as promised). **Answer: D**

Practice 11.

🧰 #7 Equilateral

Cube vertex distances come in three sizes: edge, face diagonal, space diagonal — an equilateral triangle must use three sides of the *same* size, and only face diagonals work (edges and space diagonals can never close up into a triangle of equal sides). Every face-diagonal triangle is a "corner slice": it connects the three neighbors of some cube vertex v, so it contains P exactly when v is adjacent to P. P has 3 adjacent vertices, giving exactly 3 equilateral triangles through P — one per neighboring corner. **Answer: D**

Practice 12.

🧰 #4 Angle sums / full turn

A central angle takes its fraction $\frac{\theta}{360^\circ}$ of a ring's area. Ring areas: inner ring (1 to 2): $4\pi - \pi = 3\pi$, all shaded. Outer ring (2 to 3): $9\pi - 4\pi = 5\pi$, shaded part $= \frac{\theta}{360} \cdot 5\pi$. Total area is 9π, so "shaded = unshaded" means shaded $= 4.5\pi$:

$$3\pi + \frac{\theta}{360} \cdot 5\pi = 4.5\pi \implies \frac{\theta}{360} = \frac{3}{10} \implies \theta = 108^\circ.$$

Answer: A

Practice 13.

🧰 #7 Equilateral #4 Angle sums

Extend the legs BA and CD: with two 60° base angles they meet at a third 60° angle — an **equilateral** triangle on base BC! So $BC = AB + AD$ (leg + top). Let $AB = DC = x$ and $AD = a$: perimeter $= x + (a + x) + x + a = 3x + 2a = 30$. For a to be a positive integer, x must be even: $x = 2, 4, 6, 8$ give $a = 12, 9, 6, 3$ ($x = 10$ makes $a = 0$, a degenerate triangle). That's 4 non-congruent trapezoids. **Answer: E**