

Lines · Lecture 06 · geo-lines

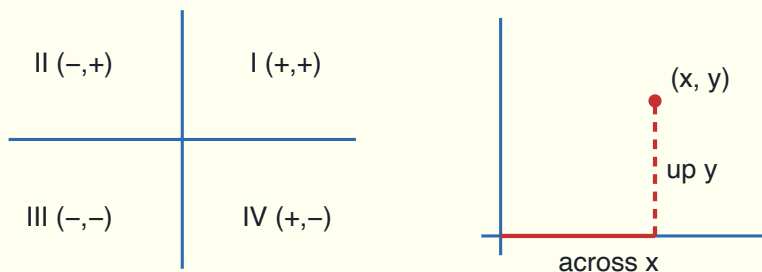
 **Course materials — Lecture 06 (93 min):** lecture video · handout PDF (Lecture 6) · board notes (06.jpg) · homework answer key (Chapter 6) — all in the `amc8/video` folder. ·  [Printable PDF](#) (answers & solutions at the back)

Key facts

Every non-vertical line is $y = mx + b$, where m is the **slope** (steepness: $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$) and b is the **y-intercept** (where it crosses the y -axis). Parallel lines share a slope; perpendicular slopes multiply to -1 . For two points: distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

Instructor's toolbox (from the lecture video & board notes)

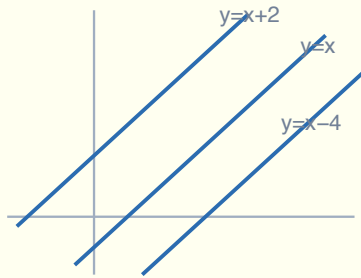
1 · Read a point like an address. (x, y) means: go *across* x , then *up* y — in that order, always (the instructor plots $(2, 1)$ and $(3, 4)$ on the board to drill it). Signs tell you the quadrant: I $(+, +)$, II $(-, +)$, III $(-, -)$, IV $(+, -)$ — so $(-1, 3)$ lives in quadrant II. A point is *on* a line exactly when its coordinates satisfy the equation.



2 · Slope is a rate — "cause → effect". The instructor's table: width $1 \rightarrow$ area 1 , $1.5 \rightarrow 1.5$, $2 \rightarrow 2 \dots$ gives $y = x$; doubling ($1 \rightarrow 2$, $2 \rightarrow 4$) gives $y = 2x$. The coefficient of x is the rate. In a **distance–time graph the slope is the speed**: constant speed = straight segment, resting = flat (slope 0), faster = steeper. That single idea solves Examples 1–2 and Practice 3 with no computation.

3 · Slope from two points. $m = \frac{y_2 - y_1}{x_2 - x_1}$ — board example: through $(0, -3)$ and $(7, 6)$, $m = \frac{6 - (-3)}{7 - 0} = \frac{9}{7}$. **Careful:** subtract in the *same order* on top and bottom — mixing orders flips the sign.

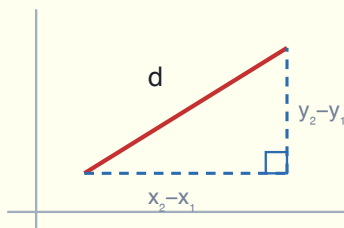
4 · $y = kx + b$ is the line's ID card. Slope k and y -intercept b can be read straight off the equation. Two special cases from the board: the horizontal line $y = 4$ is really $y = 0x + 4$ (slope 0); the vertical line $x = -6$ has **no slope** (undefined) and no $y = kx + b$ form. Same k , different b = the same line slid up or down: $y = x + 2$, $y = x$, $y = x - 4$ are three parallel copies.



5 · Parallel and perpendicular. Parallel $\Leftrightarrow$ same slope (different b). Perpendicular $\Leftrightarrow k_1 \cdot k_2 = -1$ — the board example: a line perpendicular to $y = 2x + 1$ has slope $-\frac{1}{2}$. To pin down the line, plug the known point into $y = kx + b$ and solve for b : through $(2, 1)$, $1 = -\frac{1}{2} \cdot 2 + b \Rightarrow b = 2$ (Practice 1's full solution).

6 · Intersection = solve the system; intercepts = set the other variable to 0. Where $y = x$ meets $y = -x + 6$: set them equal, $x = -x + 6 \Rightarrow x = 3$, point $(3, 3)$ — that's the whole board demo. x -intercept: set $y = 0$; y -intercept: set $x = 0$. Three lines enclosing a triangle? The vertices are the three pairwise intersections (Example 3).

7 · The distance formula is Pythagoras in disguise. The segment is a hypotenuse whose legs are the coordinate differences: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ — the board draws the right triangle under the segment ($c^2 = a^2 + b^2$, then $c = \sqrt{a^2 + b^2}$). See [Pythagorean theorem](#) for the triples that make these come out whole.



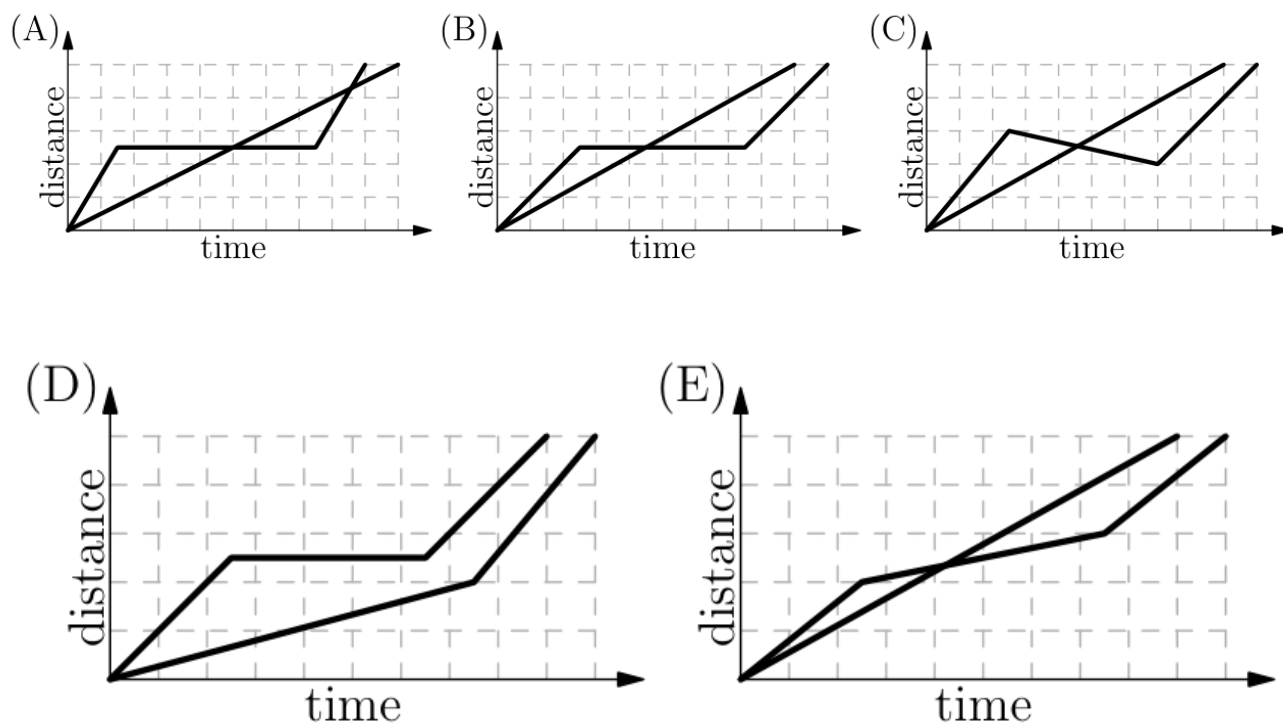
8 · Midpoint = average of the ends — and it works backwards. $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$. The board runs it in reverse: if $D(-1, 3)$ is the midpoint of $\overline{BC}$ with $B(2, -3)$, then $\frac{2 + m}{2} = -1 \Rightarrow m = -4$ and $\frac{-3 + n}{2} = 3 \Rightarrow n = 9$, so $C(-4, 9)$ — that is Example 5's entire solution.

9 · No coordinates? Build your own. When a pure-geometry figure gets messy, drop the origin on a natural corner so lengths become coordinates and intersections become systems of equations. The instructor solves Practice 11 this way ("you *could* chase similar triangles — we'll use line equations instead"): rectangle corner $D = (0, 0)$, then every line through it is two points and tip #3.

Worked examples

2019 AMC 8 #5 · slope = speed

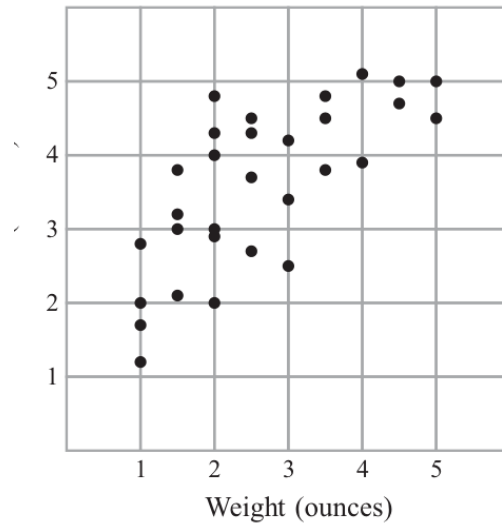
Example 1. A tortoise challenges a hare to a race. The hare eagerly agrees and quickly runs ahead, leaving the slow-moving tortoise behind. Confident that he will win, the hare stops to take a nap. Meanwhile, the tortoise walks at a slow, steady pace for the entire race. The hare awakes and runs to the finish line, only to find the tortoise already there. Which of the following graphs matches the description of the race, showing the distance d traveled by the two animals over time t from start to finish?



#2 Slope = speed

Key idea: in a distance–time graph, slope is speed. The tortoise moves at one constant speed the whole way — one straight line, never flat. The hare runs fast (steep), naps (flat, slope 0), then runs again — that shape appears only in (A) and (B). The hare arrives *later* than the tortoise, so the hare's graph must reach the finish distance at a later time: that's (B). **Answer: B**

Example 2. Laszlo went online to shop for black pepper and found thirty different black pepper options varying in weight and price, shown in the scatter plot below. In ounces, what is the weight of the pepper that offers the lowest price per ounce?



- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

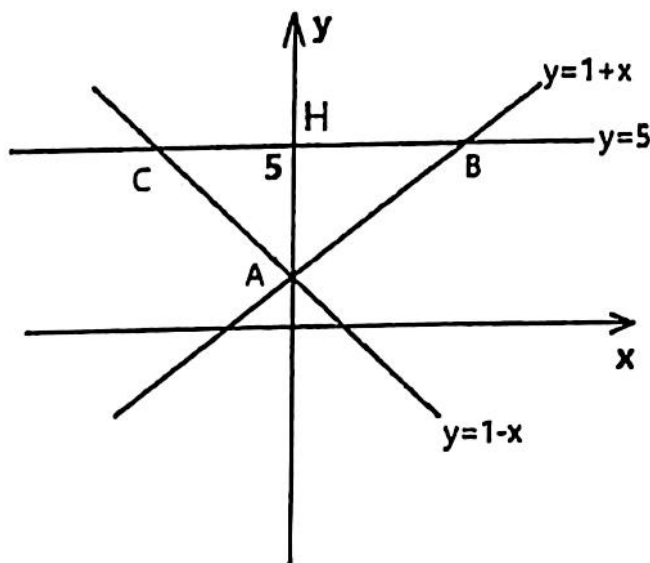


#2 Slope = rate

#3 Slope from two points

Key idea: price per ounce = $\frac{\text{price}}{\text{weight}}$ is exactly the slope of the segment from the *origin* to the point. Lowest unit price = flattest such segment: sweep a ray up from the x -axis and the first point you touch wins. That point is at weight 3 ounces (price \$2.50 — slope ≈ 0.83 dollars/oz; every other point sits above that ray). **Answer: C**

Example 3. What is the area of the triangle formed by the lines $y = 5$, $y = 1 + x$, and $y = 1 - x$?



- (A) 4 (B) 8 (C) 10 (D) 12 (E) 16



#6 Intersections

Key idea: the vertices are the pairwise intersections. $y = 1 + x$ and $y = 1 - x$: $1 + x = 1 - x \Rightarrow x = 0$, vertex $A(0, 1)$. $y = 5$ and $y = 1 + x$: $x = 4$, vertex $B(4, 5)$. $y = 5$ and $y = 1 - x$: $x = -4$, vertex $C(-4, 5)$. Base $BC = 8$ lies on $y = 5$; height from A is $5 - 1 = 4$. Area $= \frac{1}{2} \cdot 8 \cdot 4 = 16$. **Answer: E**

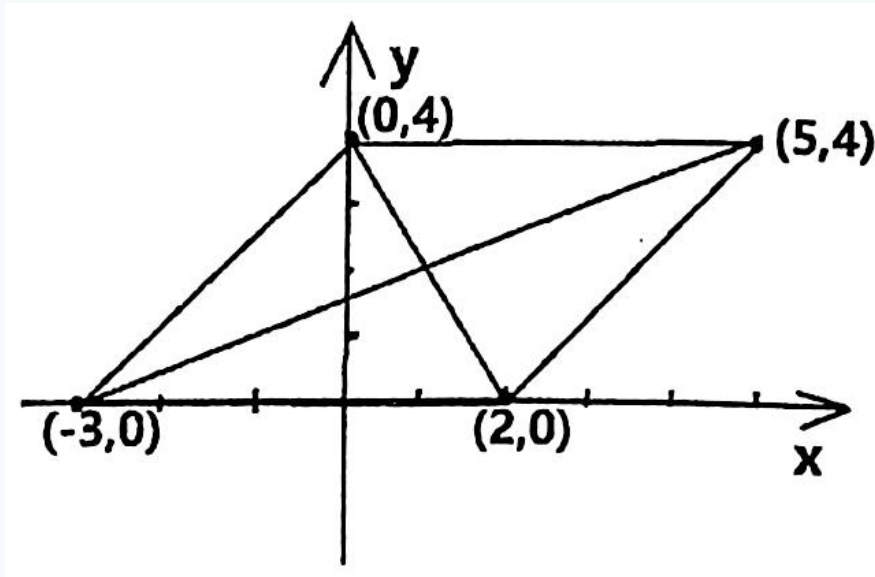
Example 4. The midpoints of the four sides of a rectangle are $(-3, 0)$, $(2, 0)$, $(5, 4)$, and $(0, 4)$. What is the area of the rectangle?

- (A) 20 (B) 25 (C) 40 (D) 50 (E) 80



#7 Distance formula

Key idea: joining the four side-midpoints of a rectangle always makes a **rhombus** (the four corner right triangles are congruent by SAS), and the rhombus's diagonals run parallel to the rectangle's sides — each diagonal *equals* a side length of the rectangle.



Diagonal 1: $(-3, 0)$ to $(5, 4)$: $\sqrt{8^2 + 4^2} = \sqrt{80} = 4\sqrt{5}$. Diagonal 2: $(2, 0)$ to $(0, 4)$: $\sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. Rectangle area $= 4\sqrt{5} \cdot 2\sqrt{5} = 40$. **Answer: C**

Example 5. Points $A(11, 9)$ and $B(2, -3)$ are vertices of $\triangle ABC$ with $AB = AC$. The altitude from A meets the opposite side at $D(-1, 3)$. What are the coordinates of point C ?

- (A) $(-8, 9)$ (B) $(-4, 8)$ (C) $(-4, 9)$ (D) $(-2, 3)$ (E) $(-1, 0)$



#8 Midpoint backwards

Isosceles three-in-one

Key idea: in an isosceles triangle the altitude from the apex is also the median — so D is the midpoint of $\overline{BC}$. Run the midpoint formula backwards: $x_C = 2x_D - x_B = -2 - 2 = -4$, $y_C = 2y_D - y_B = 6 - (-3) = 9$. So $C(-4, 9)$. **Answer: C**

Practice

Try each one before opening the solution. Figures are from the official exams and course handouts.

1. What is an equation of the line perpendicular to the line $y = 2x + 1$ that contains the point $(2, 1)$?

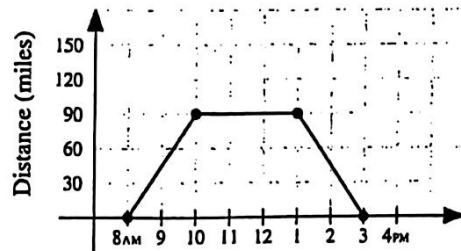
course problem · perpendicular

2. What is an equation of the line parallel to the line $y = -2x + 3$ with a y -intercept of -2 ?

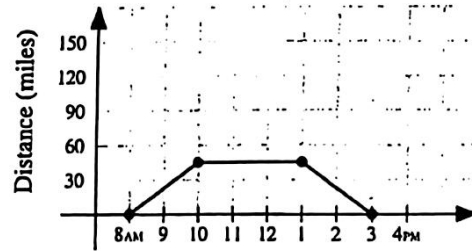
course problem · parallel

3. One sunny day, Ling decided to take a hike in the mountains. She left her house at 8 AM, drove at a constant speed of 45 miles per hour, and arrived at the hiking trail at 10 AM. After hiking for 3 hours, Ling drove home at a constant speed of 60 miles per hour. Which of the following graphs best illustrates the distance between Ling's car and her house over the course of her trip?

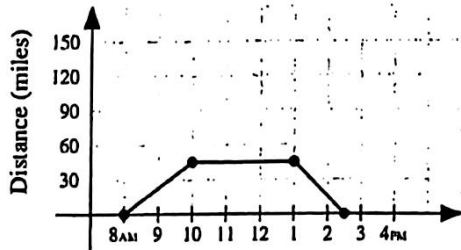
(A)



(B)



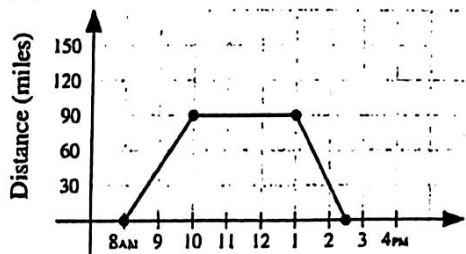
(C)



(D)



(E)



(A) (B) (C) (D) (E)

course problem · common point

4. All lines with equation $ax + by = c$ such that $a + c = 2b$ pass through a common point. What are the coordinates of that point?

(A) $(-1, 2)$ (B) $(0, 1)$ (C) $(1, -2)$ (D) $(1, 0)$ (E) $(1, 2)$

2017 AMC 10B #10 · AMC 10

5. The lines with equations $ax - 2y = c$ and $2x + by = -c$ are perpendicular and intersect at $(1, -5)$. What is c ?

- (A) -13 (B) -8 (C) 2 (D) 8 (E) 13

2003 AMC 10B #11 · AMC 10

6. A line with slope 3 intersects a line with slope 5 at the point $(10, 15)$. What is the distance between the x -intercepts of these two lines?

- (A) 2 (B) 5 (C) 7 (D) 12 (E) 20

2017 AMC 10A #12 · AMC 10

7. Let S be the set of points (x, y) in the coordinate plane such that two of the three quantities 3, $x + 2$, and $y - 4$ are equal and the third of the three quantities is no greater than this common value. Which of the following is a correct description of S ?

- (A) a single point (B) two intersecting lines
(C) three lines whose pairwise intersections are three distinct points (D) a triangle
(E) three rays with a common endpoint

2015 AMC 10B #13 · AMC 10

8. The line $12x + 5y = 60$ forms a triangle with the coordinate axes. What is the sum of the lengths of the altitudes of this triangle?

2014 AMC 10A #14 · AMC 10

9. The y -intercepts, P and Q , of two perpendicular lines intersecting at the point $A(6, 8)$ have a sum of zero. What is the area of $\triangle APQ$?

- (A) 45 (B) 48 (C) 54 (D) 60 (E) 72

2015 AMC 10A #17 · AMC 10

10. A line that passes through the origin intersects both the line $x = 1$ and the line $y = 1 + \frac{\sqrt{3}}{3}x$. The three lines create an equilateral triangle. What is the perimeter of the triangle?

- (A) $2\sqrt{6}$ (B) $2 + 2\sqrt{3}$ (C) 6 (D) $3 + 2\sqrt{3}$ (E) $6 + \frac{\sqrt{3}}{3}$

2003 AMC 10A #22 · AMC 10

11. In rectangle $ABCD$, we have $AB = 8$, $BC = 9$, H is on $\overline{BC}$ with $BH = 6$, E is on $\overline{AD}$ with $DE = 4$, line EC intersects line AH at G , and F is on line AD with $\overline{GF} \perp \overline{AF}$. Find the length of GF .

- (A) 16 (B) 20 (C) 24 (D) 28 (E) 30



Fresh from the latest exams

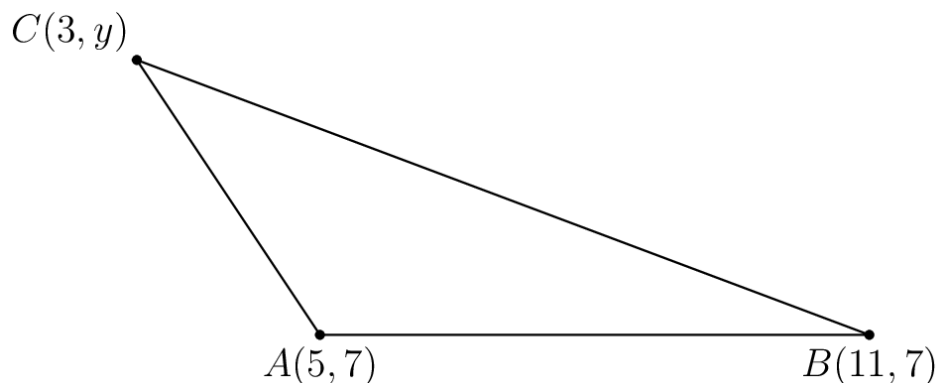
The same ideas, exactly as they appeared on the most recent tests.

2024 AMC 8 #10

12. In January 1980 the Mauna Loa Observatory recorded carbon dioxide (CO_2) levels of 338 ppm (parts per million). Over the years the average CO_2 reading has increased by about 1.515 ppm each year. What is the expected CO_2 level in ppm in January 2030? Round your answer to the nearest integer.

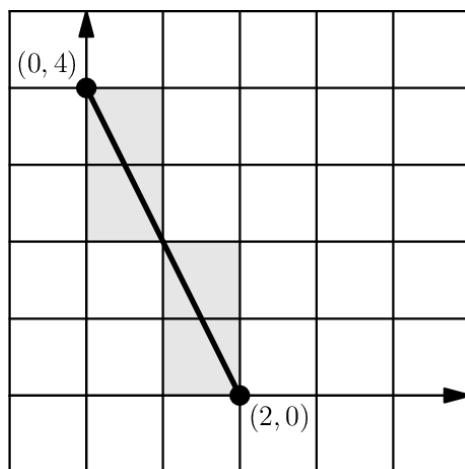
- (A) 399 (B) 414 (C) 420 (D) 444 (E) 459

13. The coordinates of $\triangle ABC$ are $A(5, 7)$, $B(11, 7)$, and $C(3, y)$, with $y > 7$. The area of $\triangle ABC$ is 12. What is the value of y ?



- (A) 8 (B) 9 (C) 10 (D) 11 (E) 12

14. Rodrigo has a very large sheet of graph paper. First he draws a line segment connecting point $(0, 4)$ to point $(2, 0)$ and colors the 4 cells whose interiors intersect the segment, as shown below. Next Rodrigo draws a line segment connecting point $(2000, 3000)$ to point $(5000, 8000)$. How many cells will he color this time?



- (A) 6000 (B) 6500 (C) 7000 (D) 7500 (E) 8000

★ Challenge (AMC 10 level)

2014 AMC 10A #18 · tilted square

C1. A square in the coordinate plane has vertices whose y -coordinates are 0, 1, 4, and 5. What is the area of the square?

- (A) 16 (B) 17 (C) 25 (D) 26 (E) 27

2013 AMC 10A #18 · equal-area cut

C2. Let points $A = (0, 0)$, $B = (1, 2)$, $C = (3, 3)$, and $D = (4, 0)$.

Quadrilateral $ABCD$ is cut into two pieces of equal area by a line passing through A . This line

intersects $\overline{CD}$ at the point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$?

- (A) 54 (B) 58 (C) 62 (D) 70 (E) 75

Sources: prep-course Lecture 06 (video + board notes + 18 worked examples, translated to English) and official AMC 8 / AMC 10 problems from the local bank (kb/content/banks/) and course handouts. Answers verified against official answer keys where available.

Answer key

Example	1	2	3	4	5									
	B	C	E	C	C									
Practice	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	$y = -\frac{1}{2}x + \frac{2}{2}$	$y = -2x - \frac{2}{2}$	E	A	E	A	E	$\frac{281}{13}$	D	D	B	B	D	C
Challenge	C1	C2												
	B	B												

Solutions to practice problems

Practice 1.



#5 Perpendicular

#4 $y=kx+b$

Perpendicular to slope 2 means slope $-\frac{1}{2}$, so $y = -\frac{1}{2}x + b$. Plug in $(2, 1)$: $1 = -\frac{1}{2} \cdot 2 + b \Rightarrow b = 2$. **Answer:** $y = -\frac{1}{2}x + 2$

Practice 2.



#5 Parallel

#4 $y=kx+b$

Parallel means the same slope, -2 ; the y -intercept is given directly, $b = -2$. Both parts of $y = kx + b$ are handed to you: $y = -2x - 2$. **Answer:** $y = -2x - 2$

Practice 3.



#2 Slope = speed

Trace the trip: 8–10 AM, distance climbs $0 \rightarrow 45 \cdot 2 = 90$ miles. 10 AM–1 PM the car is parked at the trail: flat at 90 for 3 hours. Then home at 60 mph: $90 \div 60 = 1.5$ h back to 0, a *steeper* segment down, ending at 2:30 PM. Peak at 90 (kills D), flat top of length 3 h ending at 1 PM, faster return than outbound — that's (E). **Answer:** E

Practice 4.



#1 Point on a line

A point is on the line exactly when it satisfies the equation. Rewrite the condition $a + c = 2b$ as $a \cdot (-1) + b \cdot 2 = c$ — but that is precisely $ax + by = c$ evaluated at $x = -1$, $y = 2$. So every such line passes through $(-1, 2)$. **Answer: A**

Practice 5.



#5 Perpendicular

#1 Point on a line

Slopes are $\frac{a}{2}$ and $-\frac{2}{b}$; perpendicular means $\frac{a}{2} \cdot \left(-\frac{2}{b}\right) = -1$, so $a = b$. The point $(1, -5)$ satisfies both equations: $a + 10 = c$ and $2 - 5b = -c$, i.e. $c = 5b - 2 = 5a - 2$. Then $a + 10 = 5a - 2 \Rightarrow a = 3$ and $c = 13$. **Answer: E**

Practice 6.



#6 Intercepts

#3 Slope

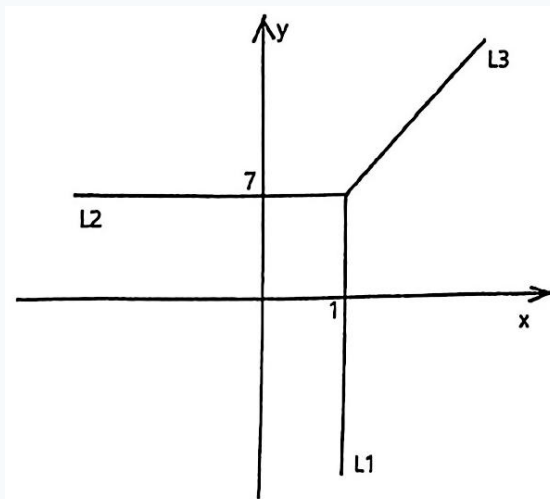
Write each line through $(10, 15)$: $y - 15 = 3(x - 10)$ and $y - 15 = 5(x - 10)$. Set $y = 0$: the first gives $x = 5$, the second $x = 7$. The x -intercepts are 2 apart. **Answer: A**

Practice 7.



#6 Intersections

Three cases for which pair is equal: (1) $3 = x + 2$ and $y - 4 \leq 3$: the ray $x = 1, y \leq 7$ going down. (2) $3 = y - 4$ and $x + 2 \leq 3$: the ray $y = 7, x \leq 1$ going left. (3) $x + 2 = y - 4$ and $3 \leq x + 2$: the ray $y = x + 6, x \geq 1$ going up-right.



All three rays start at the same point $(1, 7)$. **Answer: E**

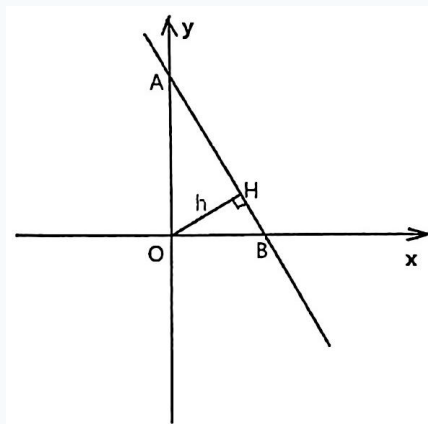
Practice 8.



#6 Intercepts

5-12-13 triple

Intercepts: $x = 5$ gives $B(5, 0)$; $y = 12$ gives $A(0, 12)$ — a right triangle with legs 5, 12 and hypotenuse 13. Two altitudes are the legs: 12 (onto OB) and 5 (onto OA). For the third, compute the area two ways: $AO \cdot BO = AB \cdot h \Rightarrow 60 = 13h \Rightarrow h = \frac{60}{13}$.



Sum $= 5 + 12 + \frac{60}{13} = \frac{281}{13}$. **Answer: $\frac{281}{13}$**

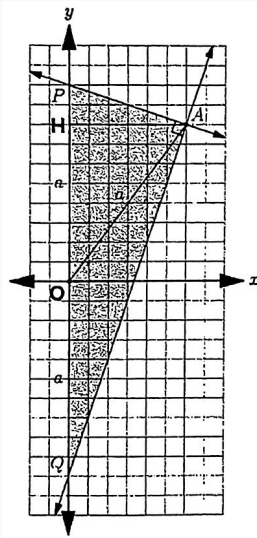
Practice 9.



#8 Midpoint

#7 Distance

P and Q sit on the y -axis with y -coordinates summing to 0 — so the origin O is the *midpoint* of $\overline{PQ}$. The lines are perpendicular at A , so $\triangle APQ$ has a right angle at A , and the median to the hypotenuse is half the hypotenuse: $PQ = 2AO$. $AO = \sqrt{6^2 + 8^2} = 10$ (a 6-8-10 triple), so $PQ = 20$. The height from A to the y -axis is just $x_A = 6$.



$$\text{Area} = \frac{1}{2} \cdot 20 \cdot 6 = 60. \text{ Answer: D}$$

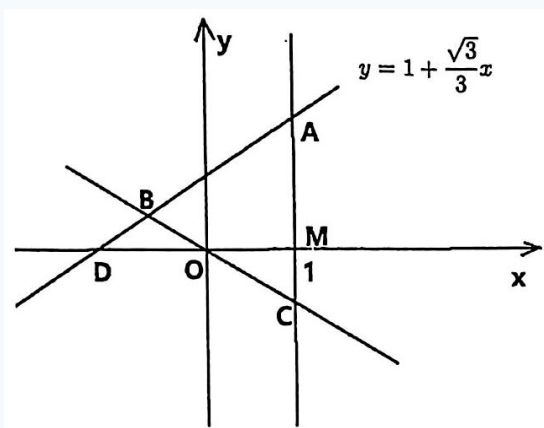
Practice 10.



#6 Intersections

30-60-90

Slope $\frac{\sqrt{3}}{3}$ means the given line rises at 30° ; the equilateral triangle stands on the vertical line $x = 1$, so its side $\overline{AC}$ is vertical.



D is the given line's x -intercept: $0 = 1 + \frac{\sqrt{3}}{3}x \Rightarrow x = -\sqrt{3}$, and $M = (1, 0)$, so $DM = 1 + \sqrt{3}$. In right $\triangle ADM$ with $\angle ADM = 30^\circ$: $AM = \frac{DM}{\sqrt{3}} = \frac{1 + \sqrt{3}}{\sqrt{3}} = 1 + \frac{\sqrt{3}}{3}$. The mirror line through the origin makes -30° : in right $\triangle OMC$, $CM = \frac{OM}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Side $AC = AM + MC = 1 + \frac{2\sqrt{3}}{3}$, so the perimeter is $3 \left(1 + \frac{2\sqrt{3}}{3} \right) = 3 + 2\sqrt{3}$. **Answer: D**

Practice 11.

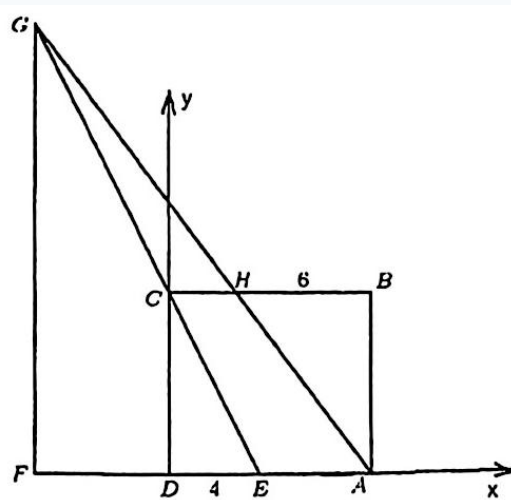


#9 Build coordinates

#3 Slope

#6 Intersections

You could chase similar triangles — instead, build coordinates: origin at D , with $\overline{DA}$ along the x -axis and $\overline{DC}$ along the y -axis. Then $C(0, 8)$, $E(4, 0)$, $A(9, 0)$, $H(3, 8)$.



Line EC : slope $\frac{0-8}{4-0} = -2$, so $y = -2(x - 4)$. Line AH : slope $\frac{8-0}{3-9} = -\frac{4}{3}$, so $y = -\frac{4}{3}(x - 9)$. Set equal: $-2x + 8 = -\frac{4}{3}x + 12 \Rightarrow -\frac{2}{3}x = 4 \Rightarrow x = -6$, $y = 20$. GF is the vertical drop from G to line AD (the x -axis): $GF = 20$. **Answer: B**

Practice 12.



#2 Slope = rate

#4 $y=kx+b$

Steady growth is a line: $y = 1.515x + 338$ with x = years since 1980. At $x = 50$: $338 + 1.515 \cdot 50 = 338 + 75.75 = 413.75 \approx 414$. **Answer: B**

Practice 13.



#1 Coordinates

$\overline{AB}$ is horizontal (both y -coordinates are 7) with length $11 - 5 = 6$ — a perfect base. The height is how far C sits above that line: $y - 7$. So $\frac{1}{2} \cdot 6 \cdot (y - 7) = 12 \Rightarrow y - 7 = 4 \Rightarrow y = 11$.

Answer: D

Practice 14.



#3 Slope from two points

The run is $\Delta x = 5000 - 2000 = 3000$ and the rise is $\Delta y = 8000 - 3000 = 5000$, so the slope is $\frac{5}{3}$. Count cells the way the pencil meets them: after the first cell, a new cell starts each time the segment crosses an interior grid line — $\Delta x - 1$ vertical ones and $\Delta y - 1$ horizontal ones — except at an interior lattice point, where both crossings coincide and only one new cell starts ($\gcd(\Delta x, \Delta y) - 1$ such points). So: $1 + (\Delta x - 1) + (\Delta y - 1) - (\gcd - 1) = \Delta x + \Delta y - \gcd(\Delta x, \Delta y)$. Check on the picture: $2 + 4 - \gcd(2, 4) = 2 + 4 - 2 = 4 \checkmark$. Here: $3000 + 5000 - \gcd(3000, 5000) = 3000 + 5000 - 1000 = 7000$. **Answer: C**

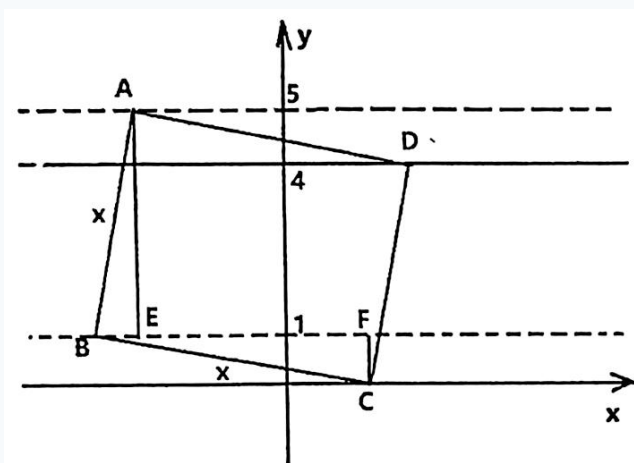
Practice C1.



#7 Distance formula

#1 Coordinates

Think of the four horizontal lines $y = 0, 1, 4, 5$. Drop perpendiculars: through A draw $AE \perp BF$ and through C draw $CF \perp BF$.



$\angle ABE + \angle CBF = 90^\circ = \angle CBF + \angle BCF$, so $\angle ABE = \angle BCF$, and $AB = BC$ gives $\triangle ABE \cong \triangle BCF$. So the side of the square is the hypotenuse of a right triangle with legs 4 (from $y = 0$ to $y = 4$) and 1 (from $y = 4$ to $y = 5$): $s^2 = 4^2 + 1^2 = 17$ — and s^2 is the area.

Answer: B

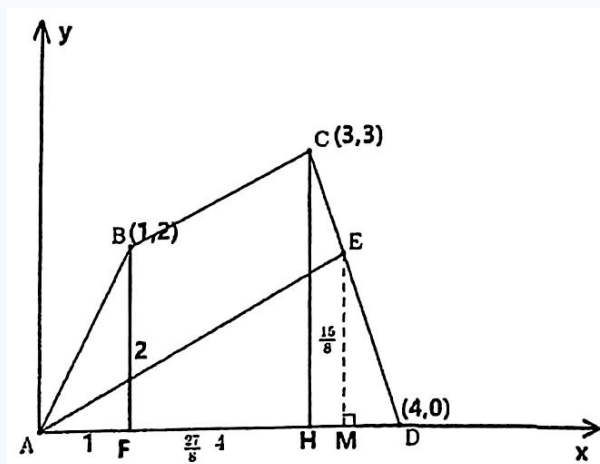
Practice C2.



#1 Coordinates

#3 Slope

First the total area, by slicing along verticals: $S_{ABCD} = 1 + \frac{(2+3)}{2} \cdot 2 + \frac{3}{2} = \frac{15}{2}$, so each piece is $\frac{15}{4}$. The cut $\overline{AE}$ (with E on $\overline{CD}$) makes $\triangle AED$ with base $AD = 4$: $\frac{1}{2} \cdot 4 \cdot y_E = \frac{15}{4} \Rightarrow y_E = \frac{15}{8}$.



Line CD drops from $(3, 3)$ to $(4, 0)$ — slope -3 . At height $\frac{15}{8}$: $x_E = 4 - \frac{15/8}{3} = 4 - \frac{5}{8} = \frac{27}{8}$. So $E = (\frac{27}{8}, \frac{15}{8})$ and $p + q + r + s = 27 + 8 + 15 + 8 = 58$. **Answer: B**