Perimeter, Area, Volume · Lecture 05 · `geo-perimeter-area-volume`

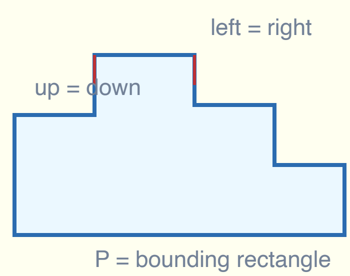
📺 **Course materials — Lecture 05 (95 min):** lecture video · handout PDF (Lecture 5) · board notes (05.jpg) · homework answer key (Chapter 5) — all in the `amc8/video` folder. · 🖨️ **Printable PDF** (answers & solutions at the back)

🔑 Key facts

The instructor's motto for this lecture: **formulas + methods**. The formulas fit on one line each — triangle $\frac{bh}{2}$, trapezoid $\frac{(a+b)h}{2}$, rhombus $\frac{d_1 d_2}{2}$, circle πr^2, any **prism** (box, cylinder, triangular prism) $V = \text{base area} \times \text{height}$. The *methods* are what the AMC actually tests: slide sides to find perimeters, subtract a small easy shape from a big easy one, count lattice points with **Pick's theorem** $S = I + \frac{B}{2} - 1$, and set up little equations when the picture won't cooperate.

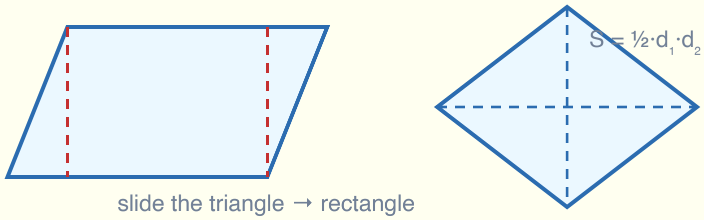
🧰 Instructor's toolbox (from the lecture video & board notes)

1 · Slide the sides: staircase perimeter = its bounding rectangle. Mark a direction on every side. The **up** pieces must total the **down** pieces, and **left** must total **right** — so all the horizontal steps slide together into one bottom edge, and the verticals into one side. The instructor's staircase: top pieces $10 + 10 + 6 = 26$, so the bottom is 26 too; each side totals 20; $P = 2(26) + 2(20) = 92$. **Careful with "chimneys":** a notch that cuts *sideways* into the shape adds extra perimeter ($2\times$ its depth) that sliding won't catch.

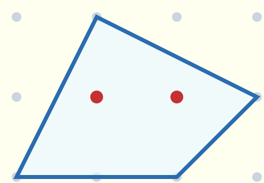


2 · Every cut adds double. Cutting a shape along an internal segment of length ℓ adds 2ℓ to the total perimeter of the pieces (both new edges are exposed). Board example: a square of side 10 cut by two full-length lines — sum of the four perimeters $= 4 \times 10 + 10 \times 2 \times 2 = 80$. Run it backwards too: gluing pieces together *hides* $2\times$ the glued edge.

3 · One formula family, all born from the rectangle. Parallelogram → cut a triangle off one end, slide it to the other: $S = bh$. Triangle = half a parallelogram: $\frac{bh}{2}$. Trapezoid: $\frac{(a+b)h}{2}$. Rhombus (diagonals $d_1 \perp d_2$): $S = \frac{d_1 d_2}{2}$ — and a square is a rhombus, so $S_{\text{square}} = \text{side}^2 = \frac{\text{diagonal}^2}{2}$. Two specials the instructor stars: right isosceles triangle with hypotenuse c: $S = \frac{c^2}{4}$; equilateral of side a : $S = \frac{\sqrt{3}}{4}a^2$.



4 · Pick's theorem — the lattice-point cheat code. For a polygon whose vertices sit on grid points: $S = I + \dfrac{B}{2} - 1$ (I = points strictly inside, B = points on the boundary). The instructor's phrasing: each interior point is worth a whole cell, each boundary point half a cell, then subtract one. Board demo: $I = 4$, $B = 12$ → $4 + 6 - 1 = 9$. **Bonus from the board:** on a *triangular* grid the count of unit triangles is $2I + B - 2$.



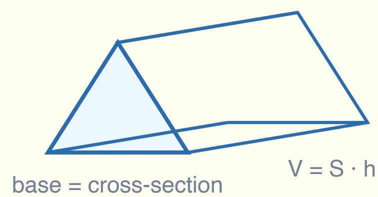
S = I + B/2 − 1

5 · Big minus small. The instructor writes "big – small" next to half the problems in this chapter: never wrestle an ugly region directly — find a big easy shape that contains it and subtract easy pieces. Tilted square in a grid → big square minus four corner triangles (Practice 4); hollow fort → full box minus the air inside (Practice 8); pentagon → big right triangle minus a rectangle (Practice 7). If the ugly region is what's *left over*, even better: it's already a difference.

6 · Cut, slide, rotate, flip — make it regular. Translate, cut & patch, rotate, reflect — the four moves of equal-area surgery that turn a weird region into a standard one. Practice 2: two quarter-circle bites refill exactly with the two halves of the bulge — the whole thing is just a rectangle. Completing the figure is the same move in reverse (Practice 6 completes a curvy region to an equilateral triangle).

7 · Name the sides, write equations. When lengths are unknown, give them letters and read the picture as equations — then look for a **combination** that answers the question directly, instead of solving for everything. Example 2: width $a+b+c=3322$, height $a-b+c=2020$; subtracting gives $2b=1302$ at once. Example 3 never finds the rectangle's sides at all — only the product $xy=18$ matters (the "whole substitution" trick, which returned on the final problem of AMC 8 2020).

8 · Prism volume = cross-section × length. Box, cube, cylinder, triangular prism — every prism is $V=S_{\text{base}}\cdot h$, where the **base is the uniform cross-section** (the face that stays the same as you slide along the solid), *not* whichever face happens to sit at the bottom. Board warning: for the tent-shaped prism, the correct "base" is the triangle $\triangle BEC$, not the rectangle underneath. Cylinder: $V=\pi r^2 h$.

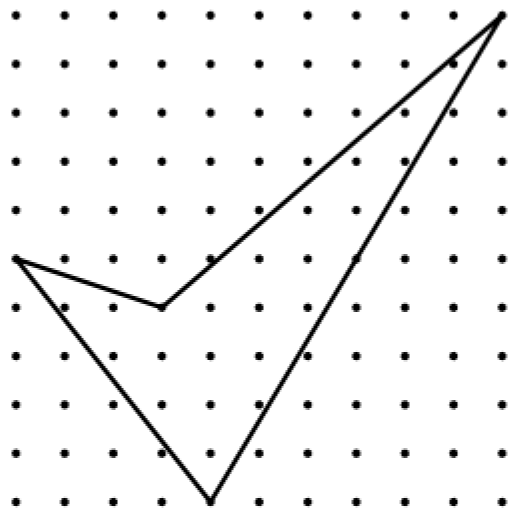


9 · Surface area by direction pairs. Look at a solid from the six directions. For solids built from boxes, **front** $=$ **back, left** $=$ **right, top** $=$ **bottom** (each pair shows the same silhouette), so compute three views and double. Steps and notches don't change a view — they only shuffle it (Practice 10: tops still total 4, the stepped sides still silhouette to the tallest piece). Gluing a cube onto a face? Only the *new side walls* add area — the new top covers exactly what it hides (Practice 11).

📖 Worked examples

2004 AMC 8 #14 · Pick's theorem

Example 1. What is the area enclosed by the geoboard quadrilateral below? (Grid points are spaced 1 unit apart.)



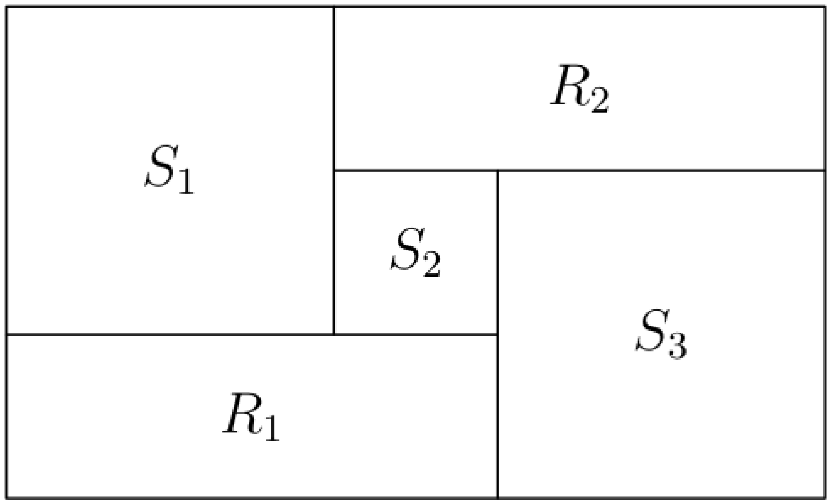
(A) 15 **(B)** $18\frac{1}{2}$ **(C)** $22\frac{1}{2}$ **(D)** 27 **(E)** 41

🧰 #4 Pick's theorem

Key idea: the vertices are on grid points, so count instead of dissecting. Boundary points: $B = 5$ (the four vertices plus one more on an edge). Interior points: $I = 21$. $S = I + \frac{B}{2} - 1 = 21 + 2\frac{1}{2} - 1 = 22\frac{1}{2}$. **Answer: C**

2020 AMC 8 #25 · equations

Example 2. Rectangles R_1 and R_2, and squares S_1, S_2, and S_3, shown below, combine to form a rectangle that is 3322 units wide and 2020 units high. What is the side length of S_2 in units?



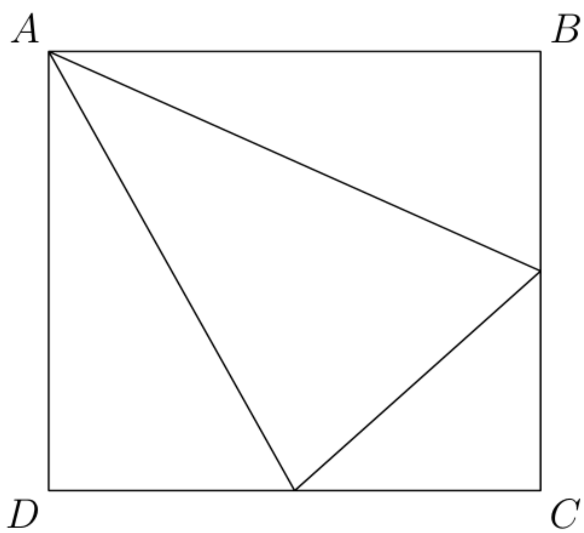
(A) 651 **(B)** 655 **(C)** 656 **(D)** 662 **(E)** 666

#7 Write equations

Key idea: let a, b, c be the side lengths of S_1, S_2, S_3. Across the middle: $a + b + c = 3322$. Vertically through S_2's column: $(a - b) + b + (c - b) = a + c - b = 2020$ (check it on the figure: the rectangles above and below S_2 have heights $a - b$ and $c - b$). Subtract: $2b = 3322 - 2020 = 1302$, so $b = 651$ — no need to find a or c at all. **Answer: A**

2000 AMC 8 #25 · whole substitution

Example 3. The area of rectangle $ABCD$ is 72 square units. If point A and the midpoints of $\overline{BC}$ and $\overline{CD}$ are joined to form a triangle, what is the area of that triangle?



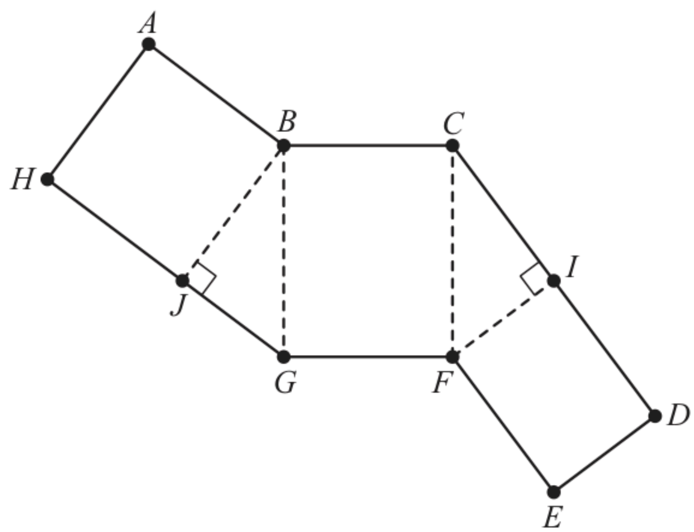
(A) 21 **(B)** 27 **(C)** 30 **(D)** 36 **(E)** 40

🧰 #7 Whole substitution #5 Big minus small

Key idea: let $CD = 2x$ and $BC = 2y$, so $4xy = 72$, i.e. $xy = 18$ — that's all we'll ever need. Subtract the three right triangles from the rectangle (with M, N the midpoints):
$S_{ABM} = \frac{1}{2} \cdot 2x \cdot y = xy = 18, \quad S_{ADN} = \frac{1}{2} \cdot 2y \cdot x = 18, \quad S_{CMN} = \frac{1}{2}xy = 9.$
Triangle $= 72 - 18 - 18 - 9 = 27$. We never learned x or y — only their product. **Answer: B**

2022 AMC 8 #24 · prism from a net

Example 4. The figure below shows a polygon $ABCDEFGH$, consisting of rectangles and right triangles. When cut out and folded on the dotted lines, the polygon forms a triangular prism. Suppose that $AH = EF = 8$ and $GH = 14$. What is the volume of the prism?



(A) 112 **(B)** 128 **(C)** 192 **(D)** 240 **(E)** 288

🧰 #8 Prism = base × length

Key idea: the two right triangles of the net must become the two bases; the three rectangles are the sides. Fold it mentally and match the edges: the triangle's legs turn out to be 8 (from AH) and $GH - 8 = 14 - 8 = 6$ (the 8 comes off GH because it glues along an edge equal to EF). Base area $= \frac{1}{2} \cdot 8 \cdot 6 = 24$; the prism's length is 8. $V = 24 \cdot 8 = 192$.

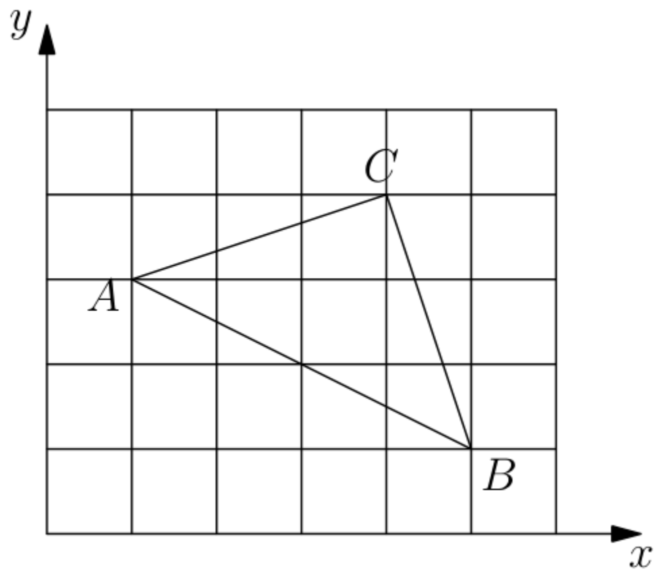
Answer: C

✏️ Practice

Try each one before opening the solution. Figures are from the official exams.

2015 AMC 8 #19

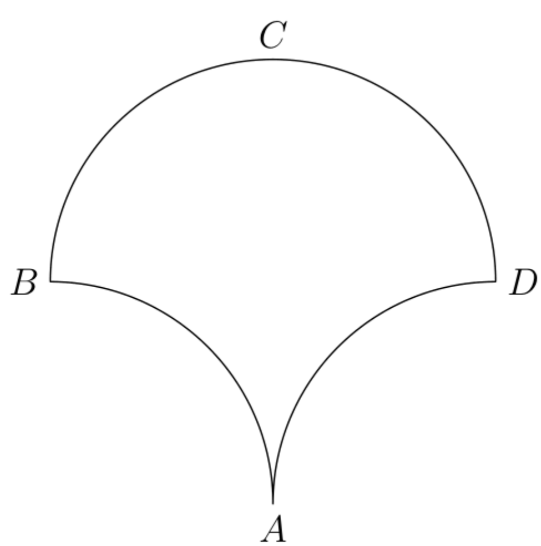
1. A triangle with vertices as $A = (1, 3)$, $B = (5, 1)$, and $C = (4, 4)$ is plotted on a 6×5 grid. What fraction of the grid is covered by the triangle?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

2000 AMC 8 #19

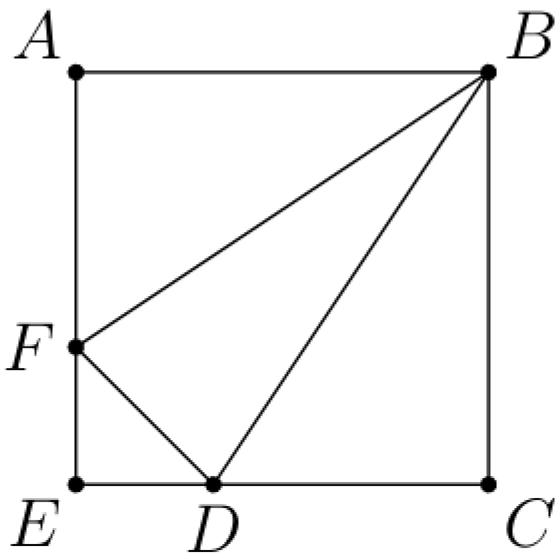
2. Three circular arcs of radius 5 units bound the region shown. Arcs AB and AD are quarter-circles, and arc BCD is a semicircle. What is the area, in square units, of the region?



(A) 25 **(B)** $10 + 5\pi$ **(C)** 50 **(D)** $50 + 5\pi$ **(E)** 25π

2008 AMC 8 #23

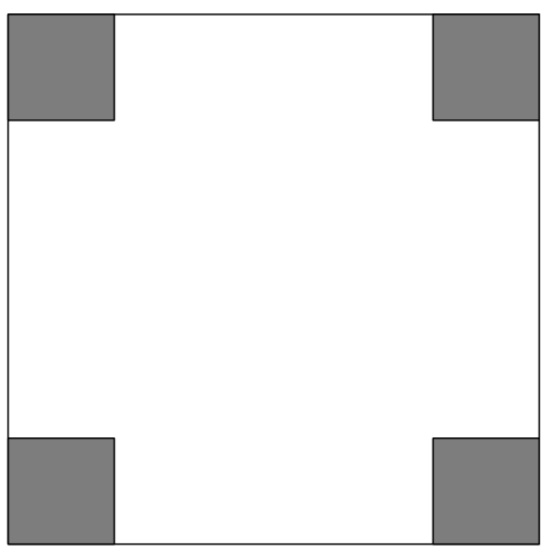
3. In square $ABCE$, $AF = 2FE$ and $CD = 2DE$. What is the ratio of the area of $\triangle BFD$ to the area of square $ABCE$?



(A) $\frac{1}{6}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{5}{18}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{7}{20}$

2015 AMC 8 #25

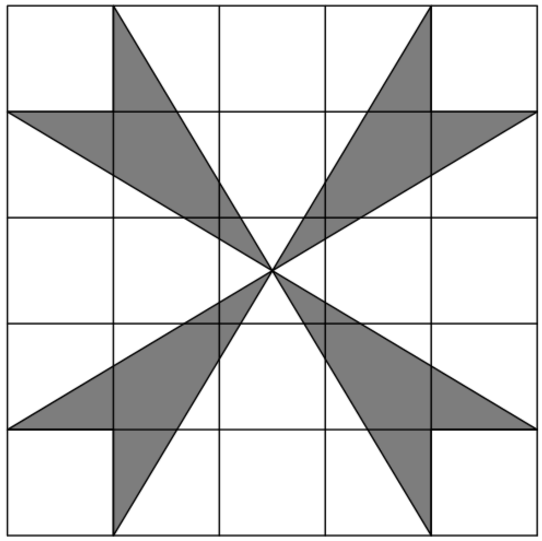
4. One-inch squares are cut from the corners of this 5 inch square. What is the area in square inches of the largest square that can fit into the remaining space?



(A) 9 **(B)** $12\frac{1}{2}$ **(C)** 15 **(D)** $15\frac{1}{2}$ **(E)** 17

2007 AMC 8 #23

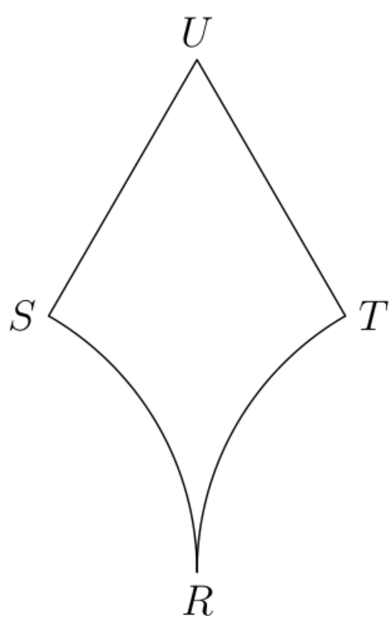
5. What is the area of the shaded pinwheel shown in the 5×5 grid?



(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

2017 AMC 8 #25

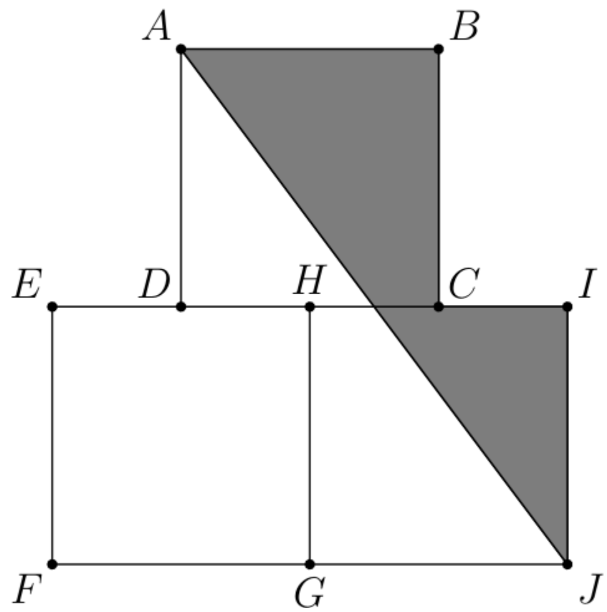
6. In the figure shown, $\overline{US}$ and $\overline{UT}$ are line segments each of length 2, and $m\angle TUS = 60^\circ$. Arcs $\overset{\frown}{TR}$ and $\overset{\frown}{SR}$ are each one-sixth of a circle with radius 2. What is the area of the region shown?



(A) $3\sqrt{3}-\pi$ **(B)** $4\sqrt{3}-\frac{4\pi}{3}$ **(C)** $2\sqrt{3}$ **(D)** $4\sqrt{3}-\frac{2\pi}{3}$ **(E)** $4+\frac{4\pi}{3}$

2013 AMC 8 #24

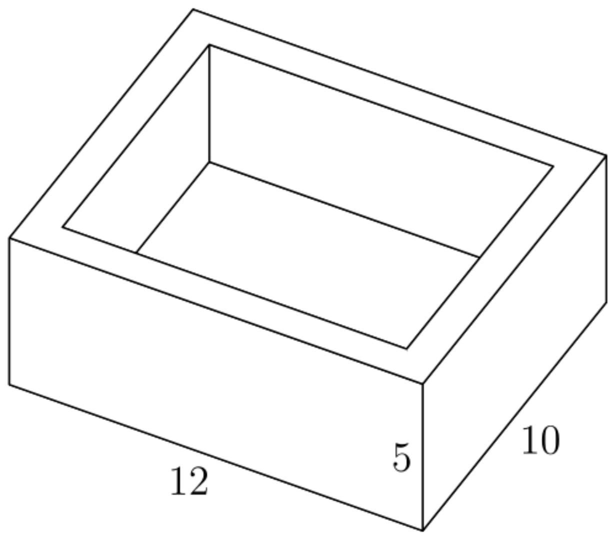
7. Squares $ABCD$, $EFGH$, and $GHIJ$ are equal in area. Points C and D are the midpoints of sides $\overline{IH}$ and $\overline{HE}$, respectively. What is the ratio of the area of the shaded pentagon $AJICB$ to the sum of the areas of the three squares?



(A) $\frac{1}{4}$ **(B)** $\frac{7}{24}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{5}{12}$

2013 AMC 8 #18

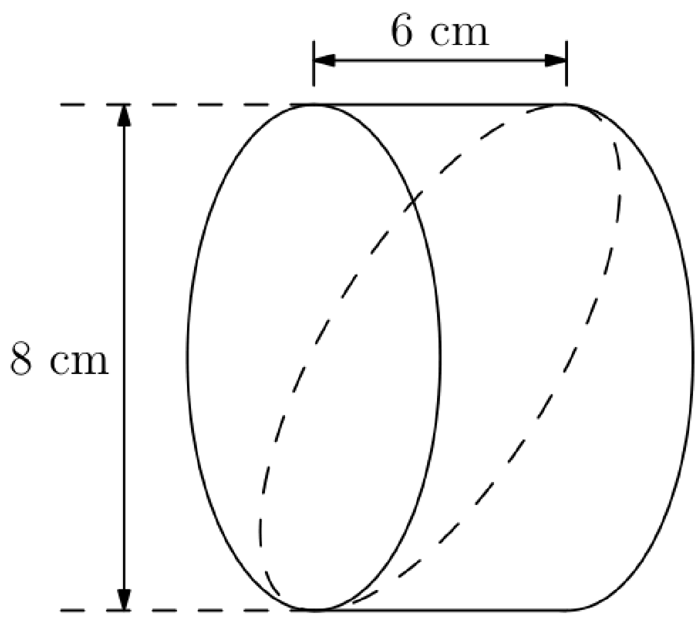
8. Isabella uses one-foot cubical blocks to build a rectangular fort that is 12 feet long, 10 feet wide, and 5 feet high. The floor and the four walls are all one foot thick. How many blocks does the fort contain?



(A) 204 **(B)** 280 **(C)** 320 **(D)** 340 **(E)** 600

2008 AMC 8 #21

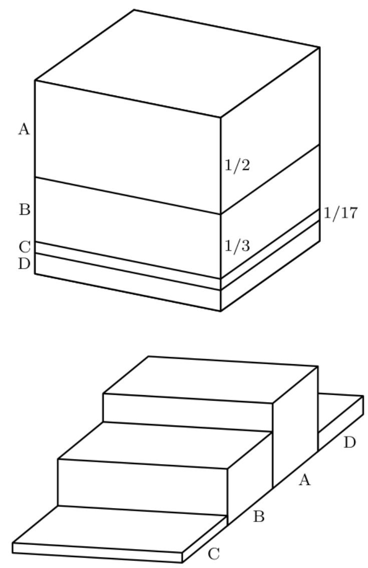
9. Jerry cuts a wedge from a 6-cm cylinder of bologna as shown by the dashed curve. Which answer choice is closest to the volume of his wedge in cubic centimeters?



(A) 48 **(B)** 75 **(C)** 151 **(D)** 192 **(E)** 603

2009 AMC 8 #25

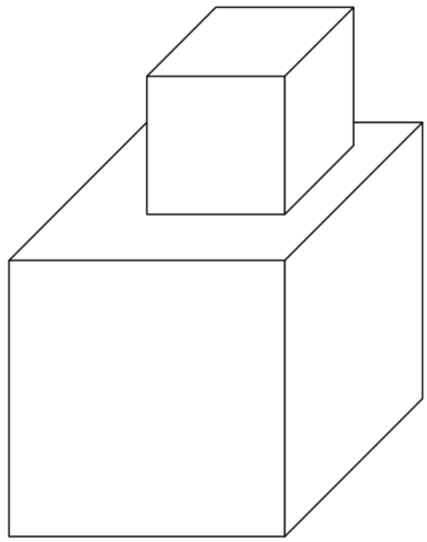
10. A one-cubic-foot cube is cut into four pieces by three cuts parallel to the top face of the cube. The first cut is $\frac{1}{2}$ foot from the top face. The second cut is $\frac{1}{3}$ foot below the first cut, and the third cut is $\frac{1}{17}$ foot below the second cut. From the top to the bottom the pieces are labeled A, B, C, and D. The pieces are then glued together end to end as shown in the second diagram. What is the total surface area of this solid in square feet?



(A) 6 **(B)** 7 **(C)** $\frac{419}{51}$ **(D)** $\frac{158}{17}$ **(E)** 11

2000 AMC 8 #22

11. A cube has edge length 2. Suppose that we glue a cube of edge length 1 on top of the big cube so that one of its faces rests entirely on the top face of the larger cube. The percent increase in the surface area (sides, top, and bottom) from the original cube to the new solid formed is closest to:



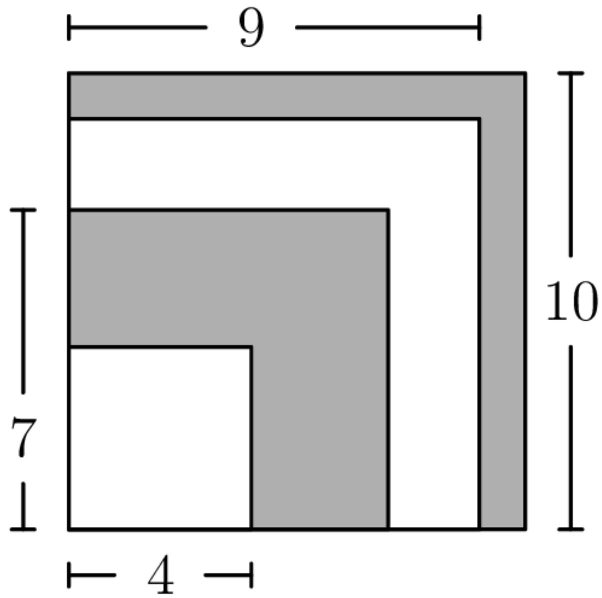
(A) 10 **(B)** 15 **(C)** 17 **(D)** 21 **(E)** 25

🆕 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2024 AMC 8 #3

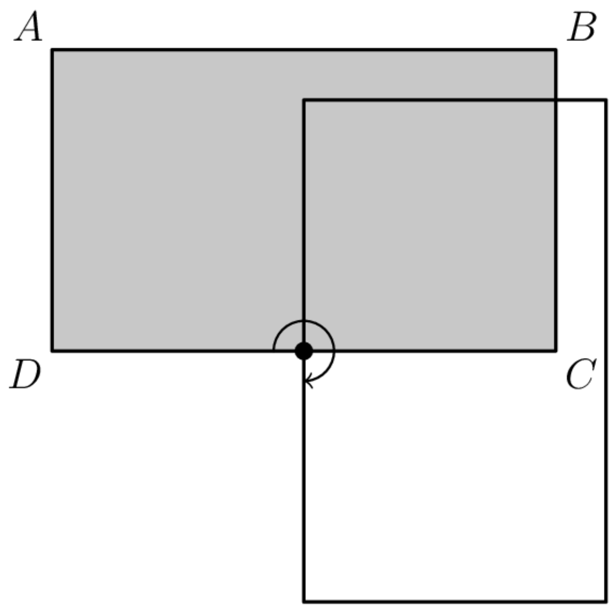
12. Four squares of side length 4, 7, 9, and 10 are arranged in increasing size order so that their left edges and bottom edges align. The squares alternate in color white-gray-white-gray, respectively, as shown in the figure. What is the area of the visible gray region in square units?



(A) 42 **(B)** 45 **(C)** 49 **(D)** 50 **(E)** 52

2025 AMC 8 #10

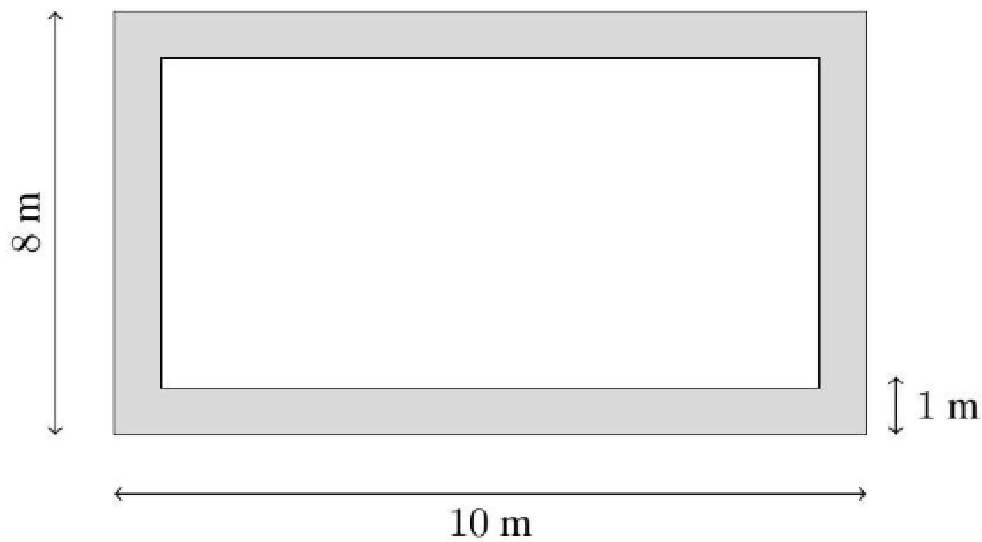
13. In the figure below, $ABCD$ is a rectangle with sides of length $AB = 5$ inches and $AD = 3$ inches. Rectangle $ABCD$ is rotated 90° clockwise around the midpoint of side $\overline{DC}$ to give a second rectangle. What is the total area, in square inches, covered by the two overlapping rectangles?



(A) 21 **(B)** 22.25 **(C)** 23 **(D)** 23.75 **(E)** 25

2026 AMC 8 #6

14. Peter lives near a rectangular field that is filled with blackberry bushes. The field is 10 meters long and 8 meters wide, and Peter can reach any blackberries that are within 1 meter of an edge of the field. The portion of the field he can reach is shaded in the figure below. What fraction of the area of the field can Peter reach?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{2}{5}$

⭐ Challenge (AMC 10 level)

2019 AMC 10B #20 · area dissection

C1. As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, $\overset{\frown}{AEB}$, $\overset{\frown}{BFC}$, and $\overset{\frown}{CGD}$, have their diameters on $\overline{AD}$, and are tangent to line EG at E, F, and G, respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form $\frac{a}{b}\pi + c - \sqrt{d}$, where a, b, c, d are positive integers and a and b are relatively prime. What is $a + b + c + d$?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

Sources: prep-course Lecture 05 (video + board notes + 16 worked examples, translated to English) and official AMC 8 problems from the local bank (`kb/content/banks/`). Answers verified against official answer keys where available.

🗝️ Answer key

Example	1	2	3	4										
	C	A	B	C										
Practice	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	A	C	C	C	B	B	C	B	C	E	C	E	D	E
Challenge	C1													
	E													

📝 Solutions to practice problems

Practice 1.

🧰 #4 Pick's theorem

All three vertices are lattice points, so Pick it: $B = 4$ (the three vertices plus $(3, 2)$, the midpoint of edge AB) and $I = 4$. $S = 4 + \frac{4}{2} - 1 = 5$. The grid is $6 \times 5 = 30$, so the fraction is $\frac{5}{30} = \frac{1}{6}$. **Answer: A**

Practice 2.

🧰 #6 Cut & refill

Draw segment BD $(= 10)$ and the 10×5 rectangle below it (the two quarter-arcs are centered at its bottom corners). Below BD: rectangle minus two quarter-circles $= 50 - \frac{25\pi}{2}$. Above BD: the semicircle $= \frac{25\pi}{2}$. The circular parts cancel exactly — the bulge refills the bites: $S = 50$. **Answer: C**

Practice 3.

🧰 #5 Big minus small

Set $FE = DE = 1$, so the square has side 3 and area 9. Don't chase $\triangle BFD$ directly — subtract the three easy right triangles around it: $S_{BFD} = 9 - \underbrace{\frac{1}{2} \cdot 2 \cdot 3}_{ABF} - \underbrace{\frac{1}{2} \cdot 1 \cdot 1}_{FED} - \underbrace{\frac{1}{2} \cdot 2 \cdot 3}_{BCD} = 9 - 3 - \frac{1}{2} - 3 = \frac{5}{2}$. Ratio $= \frac{5/2}{9} = \frac{5}{18}$. **Answer: C**

Practice 4.

🧰 #5 Big minus small #3 Formula family

The best square sits tilted, its corners touching the notches. Its area $=$ big square $-$ four corner squares $-$ four slim right triangles (legs 1 and 3): $25 - 4 \cdot 1 - 4 \cdot \frac{1 \cdot 3}{2} = 25 - 4 - 6 = 15$. **Answer: C**

Practice 5.

🧰 #3 Base × height ÷ 2

Each of the four arms is a bent dart made of **two triangles**, and every one of the 8 triangles is congruent: base $= 1$ (a unit segment along a grid line, like the one from $(1, 4)$ to $(1, 5)$) and apex at the center $(2.5, 2.5)$, so height $= 1.5$. Each triangle: $\frac{1}{2} \cdot 1 \cdot 1.5 = \frac{3}{4}$. Total: $8 \cdot \frac{3}{4} = 6$. **Answer: B**

Practice 6.

🧰 #6 Complete the figure #3 Equilateral formula

Complete the region to an equilateral triangle of side 4 — its two lower vertices are the centers of the arcs. $S_{\triangle} = \frac{\sqrt{3}}{4} \cdot 4^2 = 4\sqrt{3}$; each cut-away sector is $\frac{1}{6}$ of a radius-2 circle, and there are two: $2 \cdot \frac{1}{6}\pi \cdot 2^2 = \frac{4\pi}{3}$. Area $= 4\sqrt{3} - \frac{4\pi}{3}$. **Answer: B**

Practice 7.

🧰 #6 Complete the figure #5 Big minus small

Let each square have side 1 (total area 3). Extend $\overline{AB}$ to the right and $\overline{JI}$ upward; they meet at a point M, forming right triangle AMJ with legs $AM = \frac{3}{2}$ and $MJ = 2$. The pentagon is that triangle minus the $\frac{1}{2} \times 1$ rectangle $BMIC$: $S = \frac{1}{2} \cdot \frac{3}{2} \cdot 2 - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = 1$. Ratio $= \frac{1}{3}$. **Answer: C**

Practice 8.

🧰 #5 Big minus small #8 Box volume

Each block is 1 cubic foot, so count volume. Fill the fort solid: $12 \cdot 10 \cdot 5 = 600$. The hollow inside is a box 1 foot smaller on each side and 1 foot shallower: $10 \cdot 8 \cdot 4 = 320$. Blocks $= 600 - 320 = 280$. **Answer: B**

Practice 9.

🧰 #8 Cylinder volume

The wedge is exactly **half** the cylinder (the cut runs through the axis). Cylinder: radius $= 8/2 = 4$, length 6: $V = \pi \cdot 4^2 \cdot 6 = 96\pi$. Wedge $= 48\pi \approx 150.8 \approx 151$. **Answer: C**

Practice 10.

🧰 #9 Direction pairs

Don't touch the fractions — look from the six directions. **Top:** four 1×1 tops $= 4$; **bottom** the same $= 4$. **Front:** each piece shows a $1 \times (\text{its height})$ strip, and the four heights add back to the full edge: 1; **back** $= 1$. **Left/right:** the stepped side faces telescope — together each side shows exactly the tallest piece's cross-section: $\frac{1}{2}$ each. Total: $4 + 4 + 1 + 1 + \frac{1}{2} + \frac{1}{2} = 11$. **Answer: E**

Practice 11.

🧰 #9 Only new walls count

Original surface: $6 \cdot 2^2 = 24$. The small cube's bottom hides 1 square unit of the big top — but its own top exposes 1 back, a perfect trade. Net gain: just its 4 side faces $= 4$. Increase $= \frac{4}{24} = \frac{1}{6} \approx 17\%$. **Answer: C**

Practice 12.

🧰 #5 Big minus small

Each gray square is visible except where the next (smaller, white) square covers it: gray 7 minus white 4: $49 - 16 = 33$; gray 10 minus white 9: $100 - 81 = 19$. Total $= 33 + 19 = 52$. **Answer: E**

Practice 13.

🧰 #5 Sum minus overlap

Covered area $=$ both rectangles minus the double-counted overlap: $15 + 15 - \text{overlap}$. After the quarter turn about the midpoint M of $\overline{DC}$, the long side stands vertical: the new rectangle reaches 2.5 above and 2.5 below line DC, and extends 3 to the right of M. Its overlap with the original (2.5 of width remains to the right of M, and only the part above DC up to height 2.5 counts) is a 2.5×2.5 square: 6.25. Total $= 30 - 6.25 = 23.75$. **Answer: D**

Practice 14.

🧰 #5 Big minus small

The reachable band is the field minus the unreachable core, which is 1 meter smaller on *every* side: $(10 - 2) \times (8 - 2) = 48$. Reachable $= 80 - 48 = 32$, and $\frac{32}{80} = \frac{2}{5}$. (Classic trap: the core loses 2 from each dimension, not 1.) **Answer: E**

Practice C1.

🧰 #6 Dissect smartly #5 Big minus small #3 Sector – triangle

Don't subtract the white region — build the gray directly from three easy pieces (symmetric about line EG): **(1)** the top half of the big circle: $\frac{1}{2}\pi \cdot 4 = 2\pi$; **(2)** two side pockets between the big circle, line EG and the outer semicircles: together $2\left(2 - \frac{\pi}{2}\right) = 4 - \pi$ (each is a 2×1 rectangle minus two white quarter-circles of radius 1); **(3)** the bottom lune: big-circle sector minus triangle. The chord meets the middle semicircle where $FP = 1 = \frac{1}{2}FH$, so the half-angle is 60°: sector $= \frac{120}{360}\pi \cdot 4 = \frac{4\pi}{3}$, triangle $= \frac{1}{2} \cdot 2\sqrt{3} \cdot 1 = \sqrt{3}$; lune $= \frac{4\pi}{3} - \sqrt{3}$. Total: $2\pi + (4 - \pi) + \frac{4\pi}{3} - \sqrt{3} = \frac{7}{3}\pi + 4 - \sqrt{3}$, so $a + b + c + d = 7 + 3 + 4 + 3 = 17$.

Answer: E