

# Similar Triangles · Lecture 03 · geo-similar-triangles

 **Course materials — Lecture 03:** lecture video · handout PDF (Lecture 3) · board notes (03.jpg) · homework answer key (Chapter 3) — all in the `amc8/video` folder. ·  [Printable PDF](#) (answers & solutions at the back)

## Key facts

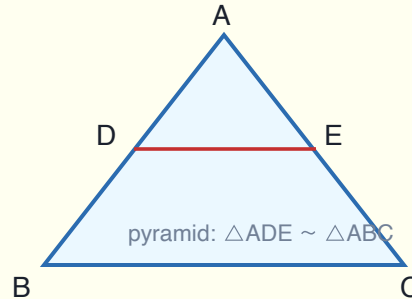
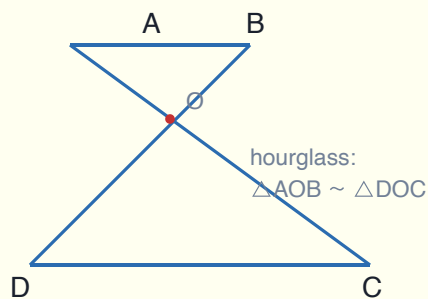
Two triangles are **similar** ( $\sim$ ) when they have the same shape at different sizes: two equal angle pairs is enough (**AA**). Every length-type quantity — sides, perimeter, heights, medians — scales by the same similarity ratio  $k$ , while **areas scale by  $k^2$** . On AMC 8 nearly every similar-triangle pair is manufactured by **parallel lines**: the "hourglass" ( $\times$  shape) and the "pyramid" (a line across a triangle).

## Instructor's toolbox (from the lecture video & board notes)

**1 · Same shape, one number  $k$ : lengths scale by  $k$ , areas by  $k^2$ .** The board's table for a 1 : 2 pair — side ratio 1 : 2, perimeter ratio 1 : 2, height ratio 1 : 2 (every *length* is the similarity ratio), but area ratio 1 : 4, "the square of the similarity ratio". Angles don't change at all — that's what "same shape" means. Going from areas back to lengths, take a square root: areas 1 : 9  $\rightarrow$  lengths 1 : 3.

**2 · AA: two angles equal  $\rightarrow$  similar.** "Angles determine the shape; sides only determine the size" — a 3-4-5 and any right triangle with the same second angle are the same picture at different zoom. The everyday case: a shared angle plus a right angle (Example 5:  $\triangle ADO \sim \triangle AOB$  share  $\angle A$ , each has a right angle).

**3 • The two parallel-line models: hourglass & pyramid.** The instructor's names for the only two pictures you'll ever see: **hourglass**:  $AB \parallel CD$  crossing at  $O \rightarrow \triangle AOB \sim \triangle DOC$  (alternate angles + vertical angle); **pyramid**:  $DE \parallel BC$  inside  $\triangle ABC \rightarrow \triangle ADE \sim \triangle ABC$  (corresponding angles). See a parallel-line pair  $\rightarrow$  name the model  $\rightarrow$  write the similarity.



**4 • Write the letters in corresponding order — the ratios then write themselves.** The teacher crossed out " $\triangle AOB \sim \triangle COD$ " on the board: in the hourglass it must be  $\triangle AOB \sim \triangle DOC$ , matching equal angles letter by letter ( $A \leftrightarrow D$ ,  $O \leftrightarrow O$ ,  $B \leftrightarrow C$ ). Once the order is right, read ratios straight across:

$$\frac{AO}{DO} = \frac{OB}{OC} = \frac{AB}{DC}.$$

Get the order wrong and every ratio you write after it is wrong too.

**5 • Part : whole.** In the pyramid model the small triangle is a scaled copy of the *whole* triangle, so

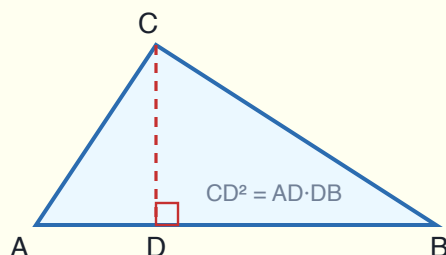
$$\frac{S_{\text{small}}}{S_{\text{whole}}} = \left( \frac{\text{near piece}}{\text{whole side}} \right)^2.$$

Example 2:  $AE : AB = 1 : 3 \rightarrow$  areas  $1 : 9$ ; Practice 1: *midline*  $\rightarrow 1 : 4$ . Complements work too: the leftover region is "whole minus similar part".

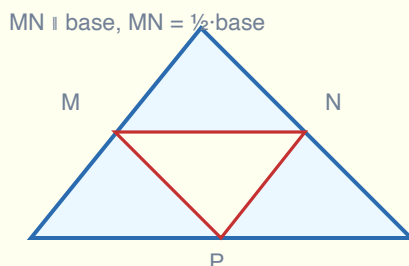
**6 · Projection theorem: the altitude to the hypotenuse makes 3 similar right triangles.** Drop  $CD \perp AB$  in right  $\triangle ACB$ : then  $\triangle ACD \sim \triangle CBD \sim \triangle ABC$ , and three products fall out:

$$CD^2 = AD \cdot DB, \quad AC^2 = AD \cdot AB, \quad BC^2 = BD \cdot AB.$$

Example 1 is one line:  $6^2 = x \cdot 4x$ . The model returns in Example 5 (radius  $\perp$  tangent) and Challenge C2 uses it twice.



**7 · Midsegment: half-size copy for free.** The segment joining the midpoints of two sides is parallel to the third side and half as long. Join all three midpoints and the midpoint triangle is similar to the original with ratio 1 : 2 — perimeter halves (Practice 3), area quarters.

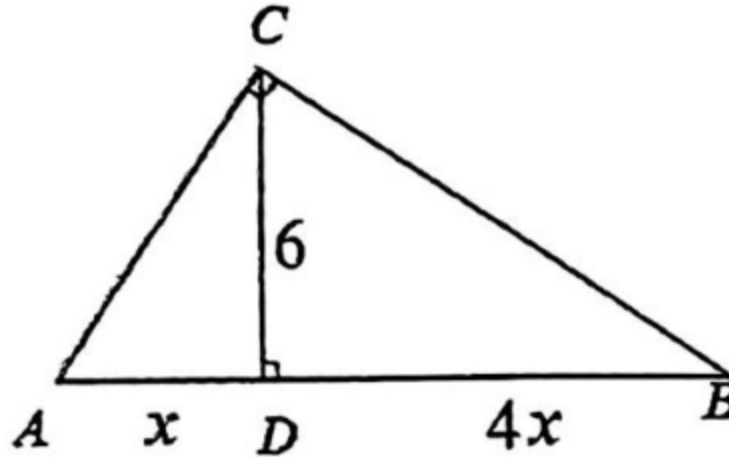


**8 · Chain area ratios — never compute a single length.** Two triangles with the *same height* have areas in the ratio of their bases (same base  $\rightarrow$  ratio of heights). Alternate these with similarity squares to walk from the known area to the target: Example 4 goes  $360 \rightarrow 240 \rightarrow 120 \rightarrow 30$  without measuring anything. And when the similar triangle you need doesn't exist yet, **draw a parallel line** to create it (Example 4's  $DG \parallel AF$ ).

## Worked examples

course mock · projection theorem

**Example 1.** In a right triangle, a perpendicular is dropped from the right angle to the hypotenuse, dividing the hypotenuse into segments of lengths  $x$  and  $4x$  inches. If the altitude is 6 inches long, then  $x$ , in inches, is:



- (A)  $\frac{1}{3}$    (B)  $\frac{2}{3}$    (C)  $\frac{3}{2}$    (D) 3   (E) cannot be determined



#6 Projection theorem

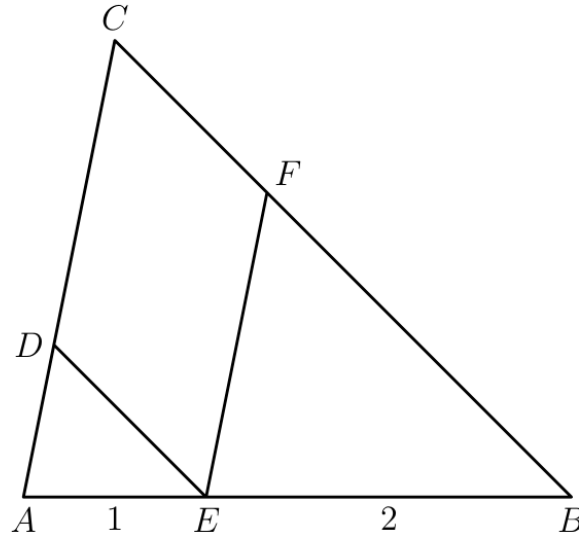
#2 AA

**Key idea:** altitude to the hypotenuse  $\rightarrow CD^2 = AD \cdot DB$  (that's just  $\triangle ACD \sim \triangle CBD$  written as a product).

$$6^2 = x \cdot 4x \implies 4x^2 = 36 \implies x = 3.$$

**Answer: D**

**Example 2.** In  $\triangle ABC$ , point  $E$  is on  $\overline{AB}$  with  $AE = 1$  and  $EB = 2$ . Point  $D$  is on  $\overline{AC}$  so that  $\overline{DE} \parallel \overline{BC}$  and point  $F$  is on  $\overline{BC}$  so that  $\overline{EF} \parallel \overline{AC}$ . What is the ratio of the area of  $CDEF$  to the area of  $\triangle ABC$ ?



- (A)  $\frac{4}{9}$    (B)  $\frac{1}{2}$    (C)  $\frac{5}{9}$    (D)  $\frac{3}{5}$    (E)  $\frac{2}{3}$



#3 Pyramid model

#5 Part : whole

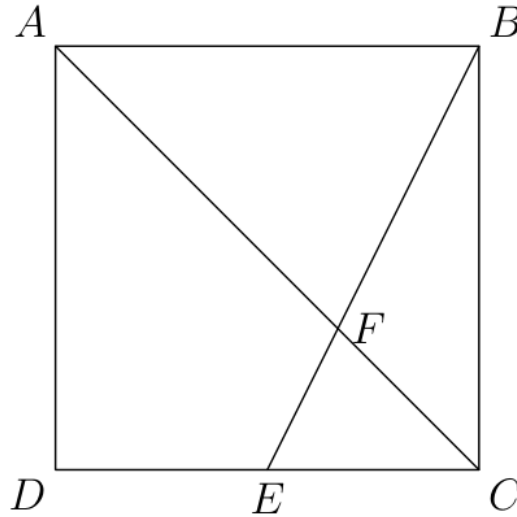
#1 Areas scale by  $k^2$ 

**Key idea:** two pyramid models cut two similar corners off the whole triangle; the parallelogram  $CDEF$  is what's left.  $DE \parallel BC$ :  $\triangle ADE \sim \triangle ABC$  with  $k = \frac{AE}{AB} = \frac{1}{3}$ , so  $\frac{S_{ADE}}{S_{ABC}} = \frac{1}{9}$ .  
 $EF \parallel AC$ :  $\triangle BEF \sim \triangle BAC$  with  $k = \frac{BE}{BA} = \frac{2}{3}$ , so  $\frac{S_{BEF}}{S_{ABC}} = \frac{4}{9}$ .

$$\frac{S_{CDEF}}{S_{ABC}} = 1 - \frac{1}{9} - \frac{4}{9} = \frac{4}{9}.$$

**Answer: A**

**Example 3.** Point  $E$  is the midpoint of side  $\overline{CD}$  in square  $ABCD$ , and  $\overline{BE}$  meets diagonal  $\overline{AC}$  at  $F$ . The area of quadrilateral  $AFED$  is 45. What is the area of  $ABCD$ ?



- (A) 100    (B) 108    (C) 120    (D) 135    (E) 144



#3 Hourglass model

#4 Correspondence order

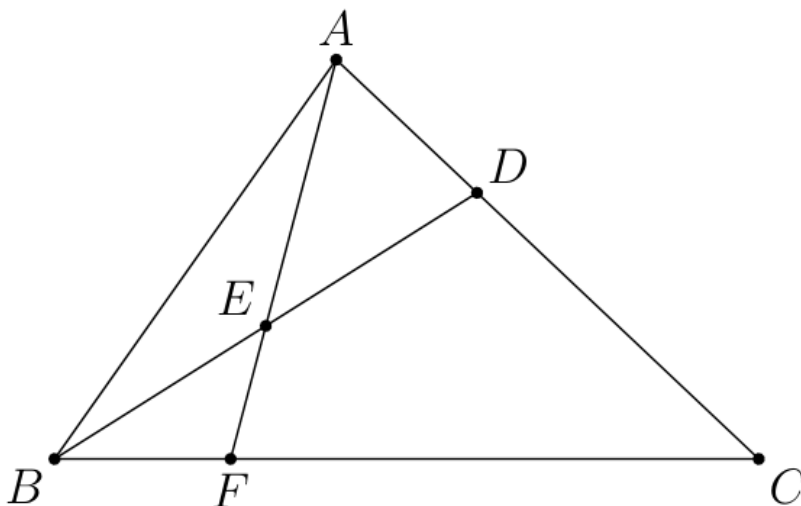
#8 Chain area ratios

**Key idea:**  $CE \parallel AB$  makes an hourglass at  $F$ :  $\triangle CEF \sim \triangle ABF$  with ratio  $CE : AB = 1 : 2$ . Set  $S_{CEF} = k$ . Areas scale by  $k^2$ :  $S_{ABF} = 4k$ . Sharing heights over  $AF$ :  $FC = 2 : 1$  (from the same hourglass):  $S_{CBF} = \frac{1}{2}S_{ABF} = 2k$ . So  $S_{ABC} = 4k + 2k = 6k$ , and the diagonal halves the square:  $S_{ACD} = 6k$  too. Quadrilateral  $AFED$  is  $\triangle ACD$  with the corner triangle  $FCE$  cut off ( $F$  on  $\overline{AC}$ ,  $E$  on  $\overline{CD}$ ):

$$S_{AFED} = S_{ACD} - S_{FCE} = 6k - k = 5k = 45 \implies k = 9.$$

The square is  $12k = 108$ . **Answer: B**

**Example 4.** In triangle  $ABC$ , point  $D$  divides side  $\overline{AC}$  so that  $AD : DC = 1 : 2$ . Let  $E$  be the midpoint of  $\overline{BD}$  and let  $F$  be the point of intersection of line  $BC$  and line  $AE$ . Given that the area of  $\triangle ABC$  is 360, what is the area of  $\triangle EBF$ ?



- (A) 24    (B) 30    (C) 32    (D) 36    (E) 40



#8 Chain area ratios

#3 Pyramid model

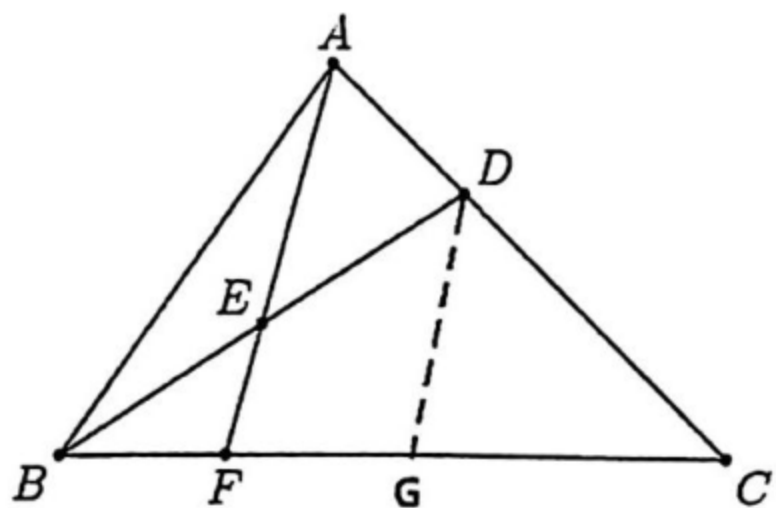
#1 Areas scale by  $k^2$ 

**Key idea:** the similar triangle you need doesn't exist — draw  $DG \parallel AF$  (meeting  $\overline{BC}$  at  $G$ ) and chain ratios down to the target. In  $\triangle CAF$ :  $DG \parallel AF$  gives  $FG : GC = AD : DC = 1 : 2$ . In  $\triangle BDG$ :  $E$  is the midpoint of  $\overline{BD}$  and  $EF \parallel DG$ , so  $BF = FG$ . Hence  $BF : FG : GC$  has  $BF = FG$  and  $FG : GC = 1 : 2$ , so  $BG : BC = 2 : 4 = 1 : 2$ . Now walk the areas:

$$S_{BDC} = \frac{DC}{AC} S_{ABC} = \frac{2}{3} \cdot 360 = 240, \quad S_{BGD} = \frac{BG}{BC} S_{BDC} = \frac{1}{2} \cdot 240 = 120,$$

and  $\triangle BEF \sim \triangle BDG$  (pyramid,  $EF \parallel DG$ ) with  $k = \frac{BE}{BD} = \frac{1}{2}$ :

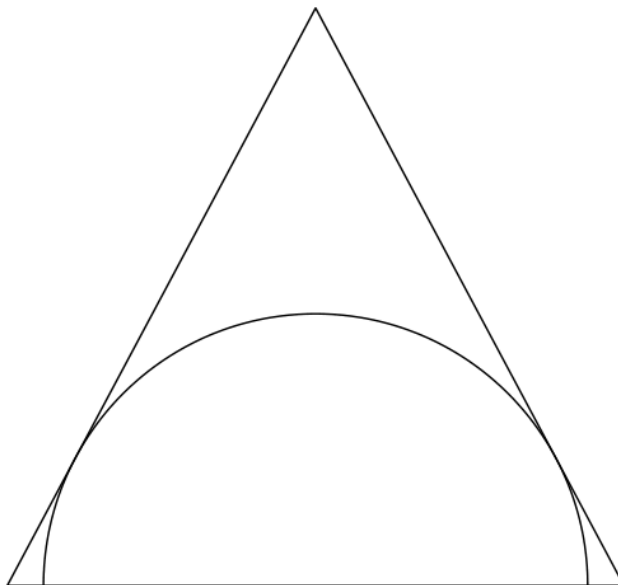
$$S_{BEF} = \frac{1}{4} \cdot 120 = 30.$$



Answer: B



**Example 5.** A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle?



- (A)  $4\sqrt{3}$    (B)  $\frac{120}{17}$    (C) 10   (D)  $\frac{17\sqrt{2}}{2}$    (E)  $\frac{17\sqrt{3}}{2}$



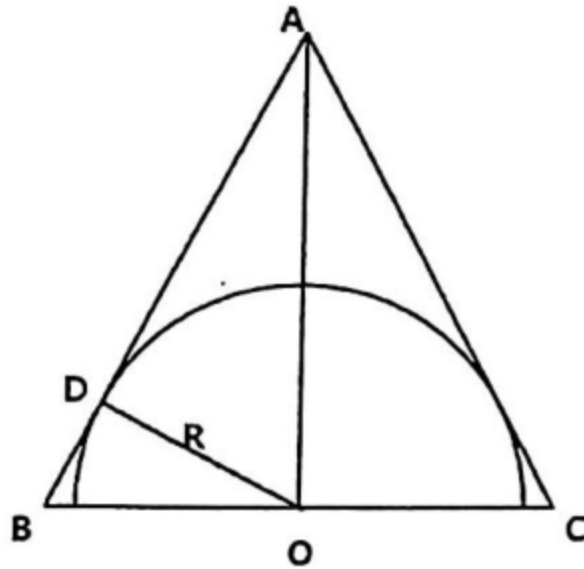
#2 AA

#6 Projection model

#4 Correspondence order

**Key idea:** the radius to the tangent point is an altitude-to-hypotenuse in disguise. Let  $O$  be the center (midpoint of base  $BC$ ) and  $D$  the tangent point on leg  $AB$ . Isosceles  $\rightarrow AO \perp BC$ , with  $BO = 8$ ,  $AO = 15$ , so  $AB = \sqrt{8^2 + 15^2} = 17$  (the 8-15-17 triple).  $OD \perp AB$  (radius  $\perp$  tangent), so  $\triangle ADO \sim \triangle AOB$  (right angle + shared  $\angle A$ ):

$$\frac{DO}{OB} = \frac{AO}{AB} \implies \frac{R}{8} = \frac{15}{17} \implies R = \frac{120}{17}.$$



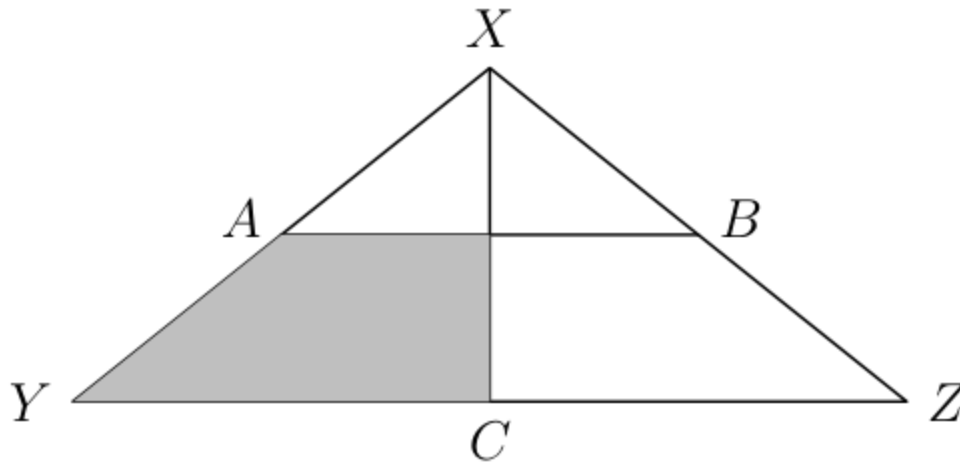
Answer: B

### Practice

Try each one before opening the solution. Figures are from the official exams and course handouts.

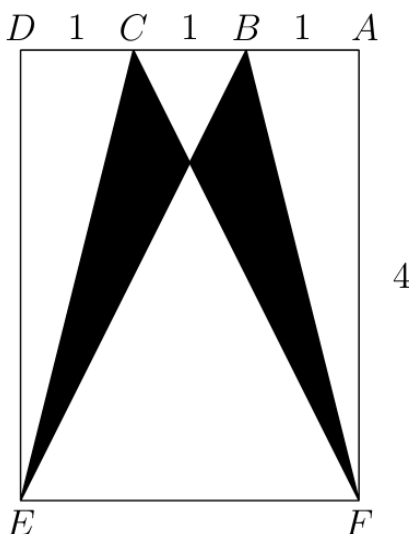
2002 AMC 8 #20

1. The area of triangle  $XYZ$  is 8 square inches. Points  $A$  and  $B$  are midpoints of congruent segments  $\overline{XY}$  and  $\overline{XZ}$ . Altitude  $\overline{XC}$  bisects  $\overline{YZ}$ . What is the area (in square inches) of the shaded region?



- (A)  $1\frac{1}{2}$    (B) 2   (C)  $2\frac{1}{2}$    (D) 3   (E)  $3\frac{1}{2}$

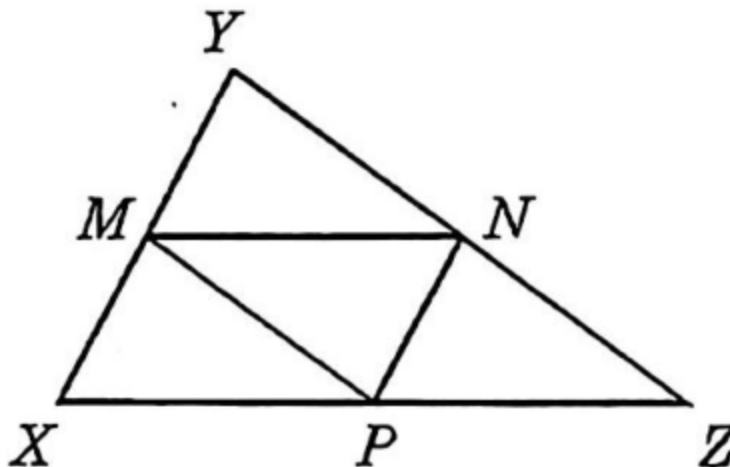
2. Rectangle  $DEFA$  below is a  $3 \times 4$  rectangle with  $DC = CB = BA$ . What is the area of the "bat wings" (shaded area)?



- (A) 2    (B)  $2\frac{1}{2}$     (C) 3    (D)  $3\frac{1}{2}$     (E) 5

3. In  $\triangle XYZ$ , points  $M$ ,  $N$ , and  $P$  are all midpoints. If  $XY = 10$ ,  $YZ = 15$ , and  $XZ = 17$ , what is the perimeter of triangle  $MNP$ ?

course mock · midsegments

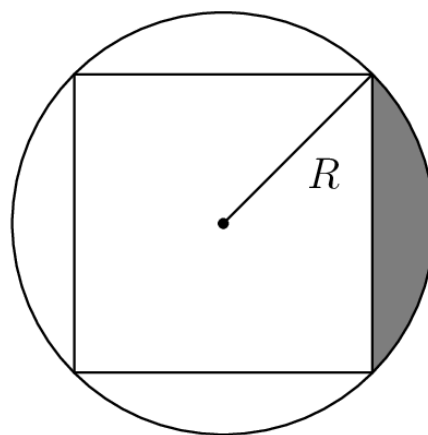
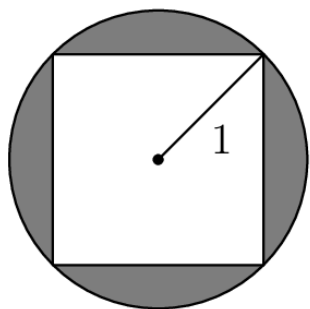


- (A) 14    (B) 16    (C)  $10\frac{2}{3}$     (D) 21    (E) cannot be determined

The same ideas, exactly as they appeared on the most recent tests.

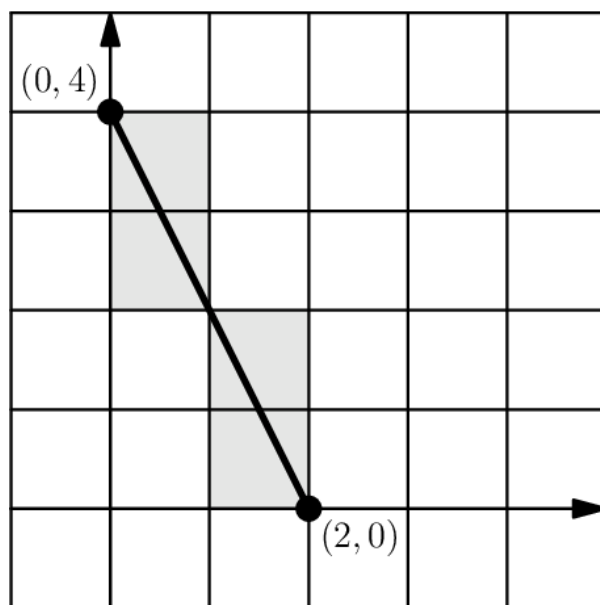
2025 AMC 8 #18

4. The circle shown below on the left has a radius of 1 unit. The region between the circle and the inscribed square is shaded. In the circle shown on the right, one quarter of the region between the circle and the inscribed square is shaded. The shaded regions in the two circles have the same area. What is the radius  $R$ , in units, of the circle on the right?



- (A)  $\sqrt{2}$    (B) 2   (C)  $2\sqrt{2}$    (D) 4   (E)  $4\sqrt{2}$

5. Rodrigo has a very large sheet of graph paper. First he draws a line segment connecting point  $(0, 4)$  to point  $(2, 0)$  and colors the 4 cells whose interiors intersect the segment, as shown below. Next Rodrigo draws a line segment connecting point  $(2000, 3000)$  to point  $(5000, 8000)$ . How many cells will he color this time?

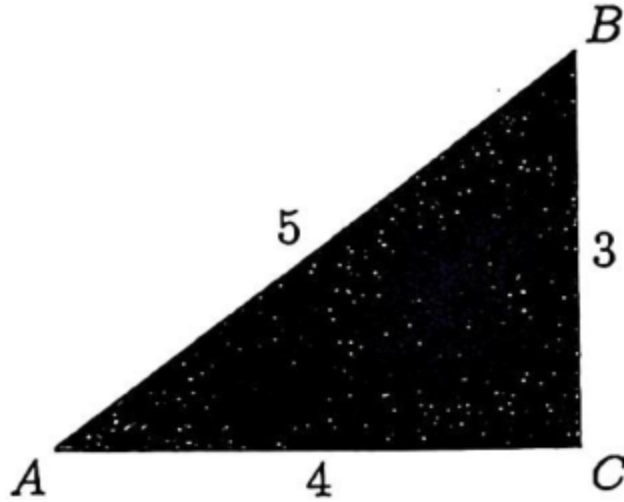


- (A) 6000    (B) 6500    (C) 7000    (D) 7500    (E) 8000

★ Challenge (AMC 10 level)

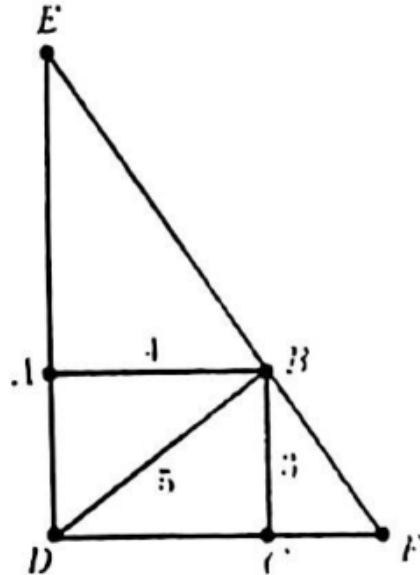
2018 AMC 10A #13 · fold = perpendicular bisector

**C1.** A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point  $A$  falls on point  $B$ . What is the length in inches of the crease?



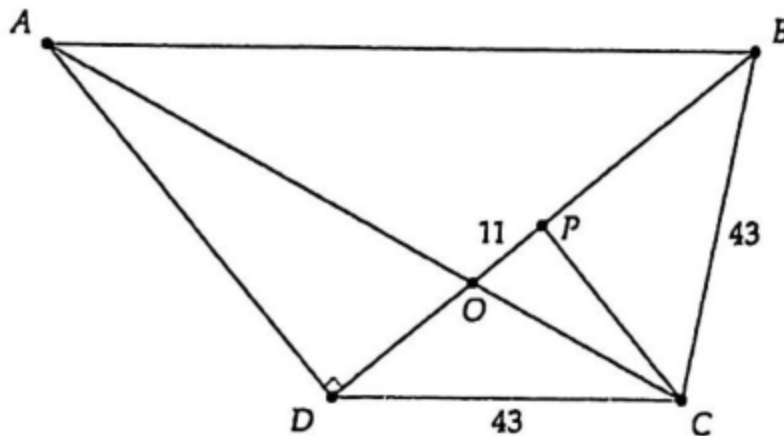
- (A)  $1 + \frac{1}{2}\sqrt{2}$    (B)  $\sqrt{3}$    (C)  $\frac{7}{4}$    (D)  $\frac{15}{8}$    (E) 2

**C2.** Rectangle  $ABCD$  has  $AB = 4$  and  $BC = 3$ . Segment  $EF$  is constructed through  $B$  so that  $EF \perp DB$ , and  $A$  and  $C$  lie on  $\overline{DE}$  and  $\overline{DF}$ , respectively. What is  $EF$ ?



- (A) 9    (B) 10    (C)  $\frac{125}{12}$     (D)  $\frac{103}{9}$     (E) 12

**C3.** Trapezoid  $ABCD$  has  $AB \parallel CD$ ,  $BC = CD = 43$ , and  $AD \perp BD$ . Let  $O$  be the intersection of the diagonals  $\overline{AC}$  and  $\overline{BD}$ , and let  $P$  be the midpoint of  $\overline{BD}$ . Given that  $OP = 11$ , the length of  $AD$  can be written in the form  $m\sqrt{n}$ , where  $m$  and  $n$  are positive integers and  $n$  is not divisible by the square of any prime. What is  $m + n$ ?



- (A) 65    (B) 132    (C) 157    (D) 194    (E) 215

Sources: prep-course Lecture 03 (video + board notes + 11 worked examples, translated to English) and official AMC 8 problems from the local bank ([kb/content/banks/](#)). Answers verified against official AoPS answer keys where available; course-mock answers follow the course solution manual.



## Answer key

|           |    |    |    |   |   |
|-----------|----|----|----|---|---|
| Example   | 1  | 2  | 3  | 4 | 5 |
|           | D  | A  | B  | B | B |
| Practice  | 1  | 2  | 3  | 4 | 5 |
|           | D  | C  | D  | B | C |
| Challenge | C1 | C2 | C3 |   |   |
|           | D  | C  | D  |   |   |

## Solutions to practice problems

### Practice 1.



#7 Midsegment

#5 Part : whole

#1 Areas scale by  $k^2$

$A$  and  $B$  are midpoints, so  $\overline{AB}$  is a midsegment:  $AB \parallel YZ$ . Let  $M$  be where  $\overline{AB}$  crosses  $\overline{XC}$ .  $S_{XYC} = \frac{1}{2} \cdot 8 = 4$  (the altitude bisects the base). Pyramid model in  $\triangle XYC$ :

$\triangle XAM \sim \triangle XYC$  with  $k = \frac{1}{2}$ , so  $S_{XAM} = \frac{1}{4} \cdot 4 = 1$ . The shaded region is the rest:

$S_{AMCY} = 4 - 1 = 3$ . **Answer: D**

### Practice 2.



#3 Hourglass model

#4 Correspondence order

#8 Chain area ratios

The two wings have equal area (symmetry), so find one. Let  $M$  be the crossing of  $\overline{CE}$  and  $\overline{BF}$  — one wing is  $\triangle ECM$ . Start from  $S_{BCE} = \frac{1}{2} \cdot CB \cdot (\text{height } 4) = \frac{1}{2} \cdot 1 \cdot 4 = 2$ .

$CB \parallel EF$  makes an hourglass:  $\triangle MCB \sim \triangle MEF$  with ratio  $CB : EF = 1 : 3$ , so the crossing point  $M$  splits the height so that  $\frac{S_{MCB}}{S_{MCE}} = \frac{BM}{ME} = \frac{1}{3}$  — i.e.  $S_{ECM} = \frac{3}{4} S_{BCE} = \frac{3}{2}$ .

One wing =  $\frac{3}{2}$ , both wings = 3. **Answer: C**

### Practice 3.



#7 Midsegment

#1 Lengths scale by k

Each side of  $\triangle MNP$  is a midsegment — parallel to a side of  $\triangle XYZ$  and half as long. So  $\triangle MNP \sim \triangle ZXY$  with ratio  $1 : 2$ , and perimeter is a *length*: it scales by  $k$ , not  $k^2$ :

$$P_{MNP} = \frac{1}{2}(10 + 15 + 17) = \frac{1}{2} \cdot 42 = 21.$$

**Answer: D**

### Practice 4.



#1 Areas scale by  $k^2$

#5 Part : whole

"Circle + inscribed square" is one shape at two zooms — every region of it scales by  $R^2$ . At radius 1: square has diagonal 2, side  $\sqrt{2}$ , area 2; ring-shaped leftover =  $\pi - 2$ . At radius  $R$  the leftover is  $(\pi - 2)R^2$ , and a quarter of it is shaded:

$$\frac{1}{4}(\pi - 2)R^2 = \pi - 2 \implies R^2 = 4 \implies R = 2.$$

**Answer: B**

### Practice 5.



#1 Same shape, scaled

Scale the segment down: it runs  $\Delta x = 3000$ ,  $\Delta y = 5000$ , and  $\gcd(3000, 5000) = 1000$ , so it is 1000 identical copies (laid corner to corner) of a segment with  $\Delta x = 3$ ,  $\Delta y = 5$  — same shape, smaller size. One  $(3, 5)$  piece never passes through an interior lattice point ( $\gcd = 1$ ), so it colors  $3 + 5 - 1 = 7$  cells (each of the 2 vertical and 4 horizontal grid lines it crosses starts a new cell). Sanity-check on the given example:  $(2, 4)$  segment  $\rightarrow 2 + 4 - 2 = 4$  cells  $\checkmark$ . Total:  $1000 \times 7 = 7000$ . **Answer: C**

### Practice C1.

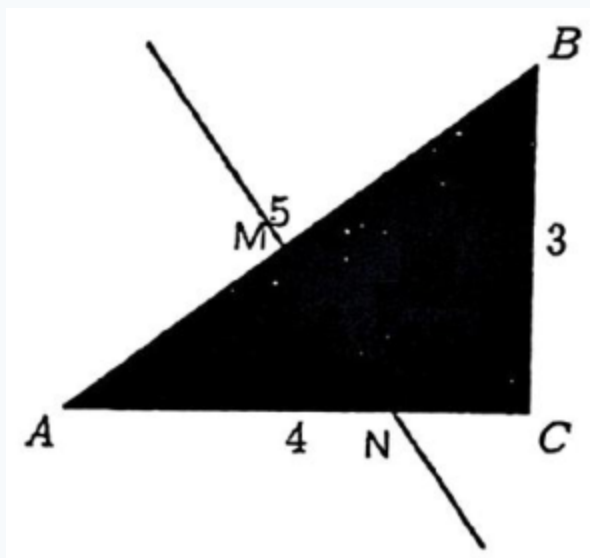


#2 AA

#4 Correspondence order

Folding  $A$  onto  $B$  means the crease is the **perpendicular bisector** of  $\overline{AB}$ : it hits  $\overline{AB}$  at its midpoint  $M$  (so  $AM = \frac{5}{2}$ ) at a right angle, and crosses  $\overline{AC}$  at  $N$ .  $\triangle AMN \sim \triangle ACB$  (right angle + shared  $\angle A$  — note the order:  $M \leftrightarrow C$ ), so

$$\frac{AM}{AC} = \frac{MN}{CB} \implies \frac{5/2}{4} = \frac{MN}{3} \implies MN = \frac{15}{8}.$$



Answer: D

### Practice C2.



#6 Projection theorem

#2 AA

$BD = 5$  (3-4-5). Use the projection theorem twice, in two different right triangles. In right  $\triangle BDF$  (right angle at  $B$ ),  $BC \perp DF$  is the altitude to the hypotenuse:  $BD^2 = DC \cdot DF \implies 25 = 4 \cdot DF \implies DF = \frac{25}{4}$ , so  $CF = \frac{25}{4} - 4 = \frac{9}{4}$ , and  $FB^2 = FC \cdot FD = \frac{9}{4} \cdot \frac{25}{4} \implies FB = \frac{15}{4}$ . In right  $\triangle EDF$  (right angle at  $D$ ),  $DB \perp EF$  is the altitude:  $BD^2 = EB \cdot BF \implies EB = \frac{25}{15/4} = \frac{20}{3}$ .

$$EF = EB + BF = \frac{20}{3} + \frac{15}{4} = \frac{80 + 45}{12} = \frac{125}{12}.$$

Answer: C

### Practice C3.



#3 Hourglass model

#4 Correspondence order

#2 AA

$CD = CB$  and  $P$  the midpoint of  $\overline{BD}$  give  $CP \perp BD$  — so  $CP$  and  $AD$  are both  $\perp BD$ , making  $\triangle CPD \sim \triangle ADB$  with ratio  $DP : DB = 1 : 2$ . Hence  $AB = 2CD = 86$ . Now the hourglass on  $AB \parallel CD$ :  $\triangle CDO \sim \triangle ABO$ , so with  $DO = x$  and  $BP = DP = 11 + x$  (so  $BO = 22 + x$ ):

$$\frac{AB}{CD} = \frac{BO}{DO} \implies \frac{86}{43} = \frac{22 + x}{x} \implies x = 22,$$

giving  $BD = 2(11 + 22) = 66$ . Finally, right  $\triangle ADB$ :  $AD = \sqrt{86^2 - 66^2} = \sqrt{3040} = 4\sqrt{190}$ , so  $m + n = 4 + 190 = 194$ . **Answer: D**