Primes · Lecture 15 · `nt-primes`

💻 **Course materials — Lecture 15:** lecture video · handout PDF (Lecture 15) · board notes (15.jpg) · homework answer key (Chapter 15) — all in the `amc8/video` folder. · 🖨️ **Printable PDF** (answers & solutions at the back)

🔑 Key facts

A **prime** has *exactly two* factors: 1 and itself. A **composite** number has three or more. 0 and 1 are **neither** prime nor composite — the instructor circles this every year. Every integer ≥ 2 breaks into primes in exactly one way (e.g. $12 = 2^2 \times 3$) — that factorization is the number's "ID card", and almost every problem in this lecture starts by reading it. There are 25 primes below 100; the first few are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$

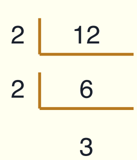
🧰 Instructor's toolbox (from the lecture video & board notes)

1 · Classify by counting factors. $13 = 1 \times 13$ and nothing else — two factors, prime. $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$ — six factors, composite. A composite number always has *at least three* factors. And $0, 1$ are neither prime nor composite — answer-choice writers love to bait you with "1 is prime".

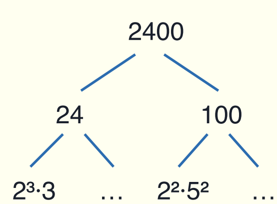
2 · Prime fingerprints. 2 is the *only even* prime; 5 is the *only* prime ending in 5; every other prime ends in $1, 3, 7$ or 9. So when a problem says **two primes add up to an odd number**, one of them must be 2 (odd $=$ odd $+$ even, and the even prime is unique) — that single observation solves Example 1.

3 · Test a prime by trial division — but only up to $\sqrt{N}$. Try the primes $2, 3, 5, 7, \ldots$ in order; you may stop as soon as the square of the next prime exceeds N. Is 101 prime? Test $2, 3, 5, 7$ and stop ($11^2 = 121 > 101$) — prime. Is 503 prime? Test primes up to 19 ($23^2 = 529 > 503$) — prime. Is 2023 prime? $45^2 = 2025$, so test primes up to 43… and $2023 = 7 \times 17^2$ — composite. The board's warning: skipping the square-root cutoff wastes minutes you don't have.

4 · Factor with short division — divide by primes only. Peel primes off one at a time and collect the powers: $2400 = 2^5 \times 3 \times 5^2$. The instructor crosses out "divide 2400 by 100" in red — chunks like 100 hide the structure; primes reveal it. Watch the wording: 2400 has **3 kinds** of prime factors $(2, 3, 5)$ but **8 prime factors counted with multiplicity**.



short division:
$12 = 2^2 \times 3$



factor tree → $2^5 \times 3 \times 5^2$

5 · Squares and cubes wear their exponents. A number is a **perfect square** exactly when every exponent in its prime factorization is *even* ($2^4 \times 5^2 \times 7^6$ ✓, $2^2 \times 5 \times 7^3$ ✗), and a **perfect cube** exactly when every exponent is a *multiple of 3*. So $4^{2019} = 2^{4038}$ is a square while 2^{2017} is not (Practice 6), and "make $360x$ a square" just means "top up each exponent to an even number" (Practice 7). Keep $2^{10} = 1024$ and the cubes $1, 8, 27, 64, 125, 216, 343$ in your pocket.

6 · The divisor-counting theorem. If $N = p^a q^b r^c \cdots$, the number of positive divisors is

$$d(N) = (a+1)(b+1)(c+1)\cdots$$

— *multiply* the (exponent$+1$)'s, never add. And memorize the tiny table: exactly 1 factor $\to$ the number 1; exactly $2 \to$ a prime; exactly $3 \to$ a prime squared $(4, 9, 25, 49, \ldots)$. This runs Example 5 and Practice 11, 13 — and the whole *Counting Divisors* handout.

7 · Factor first, ask questions later. When two numbers must share hidden structure, prime-factorize both:

- **Money problems**: $143 = 11 \times 13$ and $187 = 11 \times 17$ — the shared 11 is the unit price (Example 2);
- **Big powers**: don't compute $13^4 - 11^4$; use the difference of squares twice: $13^4 - 11^4 = (13^2 + 11^2)(13^2 - 11^2) = 290 \times 24 \times 2$ — now the powers of 2 are visible (Example 4);
- **Digit products**: $120 = 2^3 \times 3 \times 5$ tells you exactly which digits 1–9 can build it (Example 3).

8 · Parity is the fastest filter. odd $\pm$ odd $=$ even; odd $\pm$ even $=$ odd; odd $\times$ odd $=$ odd; anything $\times$ even $=$ even. Also m^2 has the *same parity* as m. Run the options through the parity filter before doing any real work — Practice 5, 9, 10 fall to it alone.

9 · Consecutive integers: use the middle. An odd count of consecutive integers sums to (count) $\times$ (middle term): the board's "$x-2d,\ x-d,\ x,\ x+d,\ x+2d$ adds to $5x$". Five consecutive integers summing to 35 ? The middle is 7 — no equations needed (Practice 3).

📖 Worked examples

2014 AMC 8 #4 · only even prime

Example 1. The sum of two prime numbers is 85. What is the product of these two prime numbers?

(A) 85 **(B)** 91 **(C)** 115 **(D)** 133 **(E)** 166

🧰 #2 Fingerprints #8 Parity

Key idea: 85 is odd, and odd $=$ odd $+$ even — so one of the two primes is even. The only even prime is 2, forcing the other to be $85 - 2 = 83$ (prime ✓). Product: $2 \times 83 = 166$. **Answer: E**

2012 AMC 8 #13 · prime factorization

Example 2. Jamar bought some pencils costing more than a penny each at the school bookstore and paid $\$1.43$. Sharona bought some of the same pencils and paid $\$1.87$. How many more pencils did Sharona buy than Jamar?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

#7 Factor first #3 Trial division

Key idea: the unit price (in cents, an integer > 1) divides both totals, so factor them: $143 = 11 \times 13$ and $187 = 11 \times 17$. The only shared factor bigger than 1 is 11 — the price per pencil. So Jamar bought 13, Sharona bought 17, and $17 - 13 = 4$. **Answer: C**

2018 AMC 8 #14 · digit product

Example 3. Let N be the greatest five-digit number whose digits have a product of 120. What is the sum of the digits of N?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 20

#7 Factor first #4 Short division

Key idea: $120 = 2^3 \times 3 \times 5$, and each digit must be 1–9. To make N as large as possible, make its *leading* digit as large as possible: the biggest single digit you can assemble from $2, 2, 2, 3, 5$ is $8 = 2^3$. That leaves 5 and 3, and pad with 1s: $120 = 8 \times 5 \times 3 \times 1 \times 1$, so $N = 85311$. Digit sum: $8 + 5 + 3 + 1 + 1 = 18$. **Answer: D**

2016 AMC 8 #15 · difference of squares

Example 4. What is the largest power of 2 that is a divisor of $13^4 - 11^4$?

(A) 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 128

#7 Factor first #5 Exponents

Key idea: never compute 13^4. Factor with the difference of squares, twice:

$$13^4 - 11^4 = (13^2 + 11^2)(13^2 - 11^2) = (169 + 121)(13 + 11)(13 - 11) = 290 \times 24 \times 2.$$

Now read off the 2s: $290 = 2 \cdot 5 \cdot 29,\ \ 24 = 2^3 \cdot 3$, and the last factor is 2 — five twos in all, so the largest power of 2 dividing the number is $2^5 = 32$. **Answer: C**

2020 AMC 8 #17 · divisor counting

Example 5. How many factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6,$ and 12.)

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

🧰 #6 Divisor counting #1 Count factors

Key idea: $2020 = 2^2 \times 5 \times 101$, so it has $(2+1)(1+1)(1+1) = 12$ divisors. Counting the ones with *more* than 3 factors directly is risky — count the opposite instead. "3 or fewer factors" means: the number 1, a prime, or a prime squared. In the divisor list that's 1; the primes $2, 5, 101$; and $4 = 2^2$ — five of them. Everything else qualifies: $12 - 5 = 7$. **Answer: B**

✏️ Practice

Try each one before opening the solution.

2010 AMC 8 #14

1. What is the sum of the prime factors of 2010?

(A) 67 **(B)** 75 **(C)** 77 **(D)** 201 **(E)** 210

2014 AMC 8 #23

2. Three members of the Euclid Middle School girls' softball team had the following conversation.

Ashley: I just realized that our uniform numbers are all 2-digit primes.

Bethany: And the sum of your two uniform numbers is the date of my birthday earlier this month.

Caitlin: That's funny. The sum of your two uniform numbers is the date of my birthday later this month.

Ashley: And the sum of your two uniform numbers is today's date.

What number does Caitlin wear?

(A) 11 **(B)** 13 **(C)** 17 **(D)** 19 **(E)** 23

2015 AMC 8 #23

3. Tom has twelve slips of paper which he wants to put into five cups labeled A, B, C, D, E. He wants the sum of the numbers on the slips in each cup to be an integer. Furthermore, he wants the five integers to be consecutive and increasing from A to E. The numbers on the papers are $2, 2, 2, 2.5, 2.5, 3, 3, 3, 3, 3.5, 4,$ and 4.5. If a slip with 2 goes into cup E and a slip with 3 goes into cup B, then the slip with 3.5 must go into what cup?

(A) A **(B)** B **(C)** C **(D)** D **(E)** E

2019 AMC 8 #19

4. In a tournament there are six teams that play each other twice. A team earns 3 points for a win, 1 point for a draw, and 0 points for a loss. After all the games have been played it turns out that the top three teams earned the same number of total points. What is the greatest possible number of total points for each of the top three teams?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 26 **(E)** 30

2010 AMC 8 #22

5. The hundreds digit of a three-digit number is 2 more than the units digit. The digits of the three-digit number are reversed, and the result is subtracted from the original three-digit number. What is the units digit of the result?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

2016 AMC 8 #7

6. Which of the following numbers is not a perfect square?

(A) 1^{2016} **(B)** 2^{2017} **(C)** 3^{2018} **(D)** 4^{2019} **(E)** 5^{2020}

2009 AMC 8 #17

7. The positive integers x and y are the two smallest positive integers for which the product of 360 and x is a square and the product of 360 and y is a cube. What is the sum of x and y?

(A) 80 **(B)** 85 **(C)** 115 **(D)** 165 **(E)** 610

2018 AMC 8 #25

8. How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?

(A) 4 **(B)** 9 **(C)** 10 **(D)** 57 **(E)** 58

2005 AMC 8 #8

9. Suppose m and n are positive odd integers. Which of the following must also be an odd integer?

(A) $m+3n$ **(B)** $3m-n$ **(C)** $3m^2+3n^2$ **(D)** $(nm+3)^2$ **(E)** $3mn$

2014 AMC 8 #13

10. If n and m are integers and n^2+m^2 is even, which of the following is impossible?

(A) n and m are even **(B)** n and m are odd **(C)** $n+m$ is even **(D)** $n+m$ is odd

(E) none of these are impossible

2018 AMC 8 #18

11. How many positive factors does $23{,}232$ have?

(A) 9 **(B)** 12 **(C)** 28 **(D)** 36 **(E)** 42

2017 AMC 8 #23

12. Each day for four days, Linda traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by 5 minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of miles for the four trips?

(A) 10 **(B)** 15 **(C)** 25 **(D)** 50 **(E)** 82

2015 AMC 8 #22

13. On June 1, a group of students are standing in rows, with 15 students in each row. On June 2, the same group is standing with all of the students in one long row. On June 3, the same group is standing with just one student in each row. On June 4, the same group is standing with 6 students in each row. This process continues through June 12 with a different number of students per row each day. However, on June 13, they cannot find a new way of organizing the students. What is the smallest possible number of students in the group?

(A) 21 **(B)** 30 **(C)** 60 **(D)** 90 **(E)** 1080

mock · course problem

14. How many 3-digit palindromes are there? (A palindrome reads the same left-to-right and right-to-left, like 121 or 737.)

(A) 30 **(B)** 60 **(C)** 90 **(D)** 120 **(E)** 180

2019 AMC 8 #13

15. A palindrome is a number that has the same value when read from left to right or from right to left. (For example, 12321 is a palindrome.) Let N be the least three-digit integer which is not a palindrome but which is the sum of three distinct two-digit palindromes. What is the sum of the digits of N?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

🆕 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2024 AMC 8 #4

16. When Yunji added all the integers from 1 to 9, she mistakenly left out a number. Her incorrect sum turned out to be a square number. What number did Yunji leave out?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

2025 AMC 8 #23

17. How many four-digit numbers have all three of the following properties? (I) The tens and ones digit are both 9. (II) The number is 1 less than a perfect square. (III) The number is the product of exactly two prime numbers.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2026 AMC 8 #18

18. In how many ways can 60 be written as the sum of two or more consecutive odd positive integers that are arranged in increasing order?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Sources: prep-course Lecture 15 (video + board notes + 20 worked examples, translated to English) and official AMC 8 problems from the local bank (`kb/content/banks/`). Answers verified against the bank's official answer keys. Companion handout: Counting Divisors.

🗝 Answer key

Example	1	2	3	4	5													
	E	C	D	C	B													
Practice	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	C	A	D	C	E	B	B	E	E	D	E	C	C	C	A	E	B	B

📝 Solutions to practice problems

Practice 1.

🧰 #4 Short division

Peel primes: $2010 = 2 \times 3 \times 5 \times 67$. Sum: $2 + 3 + 5 + 67 = 77$. **Answer: C**

Practice 2.

🧰 #2 Fingerprints #8 Parity

Every pairwise sum is a date, so each is at most 31 — the three primes must come from the smallest two-digit primes $11, 13, 17, 19$, and 19 fails ($19 + 13 = 32 > 31$). So the numbers are $11, 13, 17$ with pairwise sums $24, 28, 30$. Bethany's birthday (Ashley $+$ Caitlin) is the *earliest* date and today (Bethany $+$ Caitlin) is in between, so Caitlin $<$ Ashley $<$ Bethany — Caitlin wears the smallest number, 11. **Answer: A**

Practice 3.

🧰 #9 Middle term

All the slips together add to 35, and five consecutive integers summing to 35 have middle term $35/5 = 7$: the cup sums are $5, 6, 7, 8, 9$. Now place the 3.5: cup A (sum 5) would need a 1.5 — no such slip; cup B already has 3, and $3 + 3.5 = 6.5 > 6$; cup C would need the rest to add to 3.5 — impossible with the slips left; cup E has 2, needing another 3.5 beyond the one we placed — impossible. Cup D works: $3.5 + 2 + 2.5 = 8$ ✓. **Answer: D**

Practice 4.

#9 Distribute evenly

Let each top team beat all three bottom teams twice: 6 wins $= 18$ points each. Among themselves the three top teams play 4 games each and must stay tied — the best balanced split is win one, lose one against each rival: 2 more wins $= 6$ points. Total $18 + 6 = 24$. (All 30 points would require beating the other top teams too — then they're not tied.) **Answer: C**

Practice 5.

#8 Parity & structure

Write the number as $100a + 10b + c$ with $a = c + 2$. Reversing gives $100c + 10b + a$, and

$$(100a + 10b + c) - (100c + 10b + a) = 99(a - c) = 99 \times 2 = 198.$$

The difference is *always* 198 — units digit 8. **Answer: E**

Practice 6.

#5 Exponents

A power is a square when it can be written with an even exponent: $1^{2016} = 1$, $3^{2018} = (3^{1009})^2$, $4^{2019} = 2^{4038} = (2^{2019})^2$, $5^{2020} = (5^{1010})^2$. But 2^{2017} has an odd exponent on the prime 2 — not a square.
Answer: B

Practice 7.

#5 Exponents #4 Short division

$360 = 2^3 \times 3^2 \times 5$. For a **square**, every exponent must become even: top up $2^3 \to 2^4$ and $5 \to 5^2$, so $x = 2 \times 5 = 10$. For a **cube**, every exponent must reach a multiple of 3: $3^2 \to 3^3$ and $5 \to 5^3$, so $y = 3 \times 5^2 = 75$. Then $x + y = 85$. **Answer: B**

Practice 8.

#5 Exponents

$2^8 + 1 = 257$, and $6^3 = 216 < 257 \leq 343 = 7^3$, so the first cube in range is 7^3. At the top, $2^{18} = (2^6)^3 = 64^3$, so $64^3 < 2^{18} + 1$ is in range — the last cube is 64^3. Count: $64 - 7 + 1 = 58$. **Answer: E**

Practice 9.

🧰 #8 Parity

Run the parity filter: $m + 3n$ is odd+odd $=$ even; $3m - n$ likewise even; $3m^2 + 3n^2$ is odd+odd $=$ even; $nm + 3$ is odd+odd $=$ even, so its square is even. Only $3mn$ — a product of three odds — stays odd. **Answer: E**

Practice 10.

🧰 #8 Parity

m^2 and n^2 have the same parity as m and n. For the sum of the squares to be even, m and n are *both even* or *both odd* — and in either case $n + m$ is even. So $n + m$ odd is impossible. **Answer: D**

Practice 11.

🧰 #6 Divisor counting #4 Short division

Peel primes: $23232 = 2^6 \times 3 \times 11^2$. Divisor count: $(6+1)(1+1)(2+1) = 7 \times 2 \times 3 = 42$. **Answer: E**

Practice 12.

🧰 #6 Divisors #4 Short division

If one mile takes x minutes, the day's distance is $60/x$ miles — so $x, x+5, x+10, x+15$ must *all divide* 60. Scanning the divisors of 60: only $5, 10, 15, 20$ works ($x = 5$). Distances: $12 + 6 + 4 + 3 = 25$ miles. **Answer: C**

Practice 13.

🧰 #6 Divisor counting

Twelve days of different row sizes means the total has exactly 12 divisors. Check the choices with the theorem: $21 = 3 \times 7$ gives 4; $30 = 2 \times 3 \times 5$ gives 8; $60 = 2^2 \times 3 \times 5$ gives $(2+1)(1+1)(1+1) = 12$ ✓ — and it's the smallest choice that works (both 15 and 6 do divide 60 ✓). **Answer: C**

Practice 14.

🧰 #1 Count structure

A 3-digit palindrome is $\overline{aba}$: the first and last digits match and can't be 0 — 9 choices — and the middle digit is free — 10 choices. $9 \times 10 = 90$. **Answer: C**

Practice 15.

🧰 #7 Factor first

Two-digit palindromes are exactly $11, 22, \ldots, 99$ — all multiples of 11, so the sum of three of them is a multiple of 11 too. Hunting upward from 100: the multiples $110 = 11 + 22 + 77$ works and isn't a palindrome ✓ (the smaller sums like $66, 77, \ldots, 99$ are palindromes or too small). So $N = 110$ and its digit sum is $1 + 1 + 0 = 2$. **Answer: A**

Practice 16.

🧰 #5 Squares #9 Middle term

$1 + 2 + \cdots + 9 = 45$. Leaving out x gives $45 - x$, which must be a perfect square with $36 \leq 45 - x \leq 44$ — only 36 fits, so $x = 9$. **Answer: E**

Practice 17.

🧰 #7 Factor first #3 Trial division

If $N + 1 = k^2$ and N ends in 99, then k^2 ends in 00, so k is a multiple of 10. Four-digit range: $k = 40, 50, 60, 70, 80, 90$. Factor $N = k^2 - 1 = (k-1)(k+1)$ — it's a product of exactly two primes only when *both* neighbors are prime: 39×41 ✗, 49×51 ✗, 59×61 ✓, 69×71 ✗, 79×81 ✗, 89×91 ✗ ($91 = 7 \times 13$!). Just one: $3599 = 59 \times 61$. **Answer: B**

Practice 18.

🧰 #9 Middle term #8 Parity

j consecutive odd numbers starting at a add to $j(a + j - 1)$, so we need $j(a + j - 1) = 60$ with $j \geq 2$, a odd and positive. For each divisor j of 60: $a = \frac{60}{j} - j + 1$ must be odd and ≥ 1. $j = 2 : a = 29$ ✓ $(29 + 31)$; $j = 3 : a = 18$ ✗; $j = 4 : a = 12$ ✗; $j = 5 : a = 8$ ✗; $j = 6 : a = 5$ ✓ $(5 + 7 + 9 + 11 + 13 + 15)$; $j = 10 : a < 0$ ✗. Two ways. **Answer: B**