Remainders & Divisibility · Lecture 17 · `nt-remainders-divisibility`

📼 **Course materials — Lecture 17:** lecture video · handout PDF (Lecture 17) · board notes (17.jpg) — all in the `amc8/video` folder. (This lecture has no homework-solutions chapter; the examples below were rebuilt from the board notes and the official problem bank.) · 🖨️ **Printable PDF** (answers & solutions at the back)

🔑 Key facts

Every division fits one template: $N = d \cdot q + r$ with $0 \leq r < d$ — dividend $=$ divisor $\times$ quotient $+$ remainder, and the remainder is always *smaller than the divisor*. "d divides N" just means $r = 0$. Two workhorses fall straight out of the template: the **units digit** of N is its remainder mod 10, and remainders repeat in a cycle of length d — so almost every remainder problem is really a problem about *finding the cycle*.

🧰 Instructor's toolbox (from the lecture video & board notes)

1 · Anatomy of a remainder. $20 \div 6 = 3$ r 2 means $20 = 6 \times 3 + 2$ — and the instructor's first warning: $20 = 6 \times 2 + 8$ is *not* a legal answer, because a remainder must be smaller than the divisor. When a problem gives you a quotient-and-remainder fact, immediately rewrite it as $N = dq + r$; the equation form is what you can compute with.

2 · How remainders scale. From the board, with $A \div B = C$ r D:

- Scale *both* dividend and divisor by k: $kA \div kB = C$ r kD — quotient unchanged, remainder scales ($20 \div 3 = 6$ r $2 \;\Rightarrow\; 40 \div 6 = 6$ r 4).
- Scale *only* the dividend by k: $kA \div B = kC$ r kD — then reduce: if $kD \geq B$, roll the excess into the quotient.

The trap the instructor flags in red: scaling the divisor *alone* wrecks both quotient and remainder — no shortcut there.

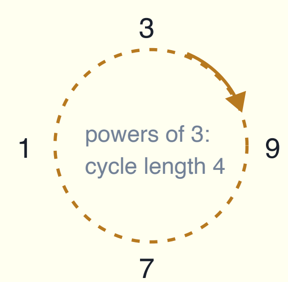
3 · The three divisibility-rule families (the board's table — knowing *why* tells you *when*):

- **Digit sum** — for 3 and 9: works because $9, 99, 999, \ldots$ are all multiples of 9;
- **Last digits** — for $2, 5$ (last digit), $4, 25$ (last two), $8, 125$ (last three): because $10 = 2 \times 5$, $100 = 4 \times 25$, $1000 = 8 \times 125$;
- **Alternating sum** — for 11: (sum of odd-position digits) $-$ (sum of even-position digits), because $9, 999, \ldots$ are one *less* and $11, 1001, \ldots$ one *more* than a power of 10; e.g. $12345 \to (5+3+1)-(4+2) = 3$, so 12345 leaves remainder 3 mod 11.

For a composite divisor, split it into coprime parts and test both: $6 = 2 \cdot 3$, $15 = 3 \cdot 5$, $12 = 3 \cdot 4$ (Example 3).

4 · The chop-and-double rule for 7. Cut off the units digit and subtract it *twice* from what's left; repeat. $3185 \to 318 - 2 \times 5 = 308 \to 30 - 2 \times 8 = 14$ — a multiple of 7, so 3185 is too. It works because $\overline{ab}$ $(= 10a + b)$ and $a - 2b$ differ by $21b$, a multiple of 7.

5 · Units digits of powers run in short cycles. Only the units digit of the base matters (2023^7 ends like 3^7), and every digit's cycle has length 1, 2 or 4. So reduce the base, find the small cycle, then divide the exponent by the cycle length and read the remainder: 13^{2012} ends like 3^{2012}, cycle $3, 9, 7, 1$, and 2012 is a multiple of 4 — units digit 1. Likewise 2024^{2024} ends like 4^{2024}, cycle $4, 6$ — units digit 6.



6 · Same remainder (or same deficit) → jump to the LCM. If N leaves the *same* remainder r when divided by several numbers, then $N - r$ is a multiple of their LCM (Practice 2: remainder 1 mod $4, 5, 6 \Rightarrow N - 1$ is a multiple of 60). If instead N falls *short by the same amount* k (remainder $=$ divisor $- k$ each time), then $N + k$ is a multiple of the LCM (Practice 9: remainders $2, 5, 7$ for $6, 9, 11$ are all "4 short" $\Rightarrow 198 \mid N + 4$). Build the LCM from prime powers, never by multiplying everything together.

7 · Repeat-block numbers factor instantly. $\overline{abcabc} = \overline{abc} \times 1001 = \overline{abc} \times 7 \times 11 \times 13$ — memorize $1001 = 7 \times 11 \times 13$. Same idea one size down: $\overline{abab} = \overline{ab} \times 101$. Any "copy-paste" number is a small number times a known constant (Example 2).

8 · Count multiples by first and last index. How many multiples of 13 are three-digit? The first is $13 \times 8 = 104$, the last is $13 \times 76 = 988$ (check: $13 \times 77 = 1001$, too big), so there are $76 - 8 + 1 = 69$. The count formula $\lfloor N/k \rfloor$ counts multiples of k up to N; the index trick avoids off-by-one errors in a range (Practice 5).

9 · Weekdays are mod 7. Calendar problems are remainder problems in disguise: step forward 10 days $=$ step forward 3 weekdays. List each event's offset mod 7 and compare against the forbidden (or required) day — that's the whole of Practice 8.

10 · Count a prime inside $n!$ with floors (Legendre). The exponent of p in $n!$ is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

Board examples: fives in $98!$: $\lfloor 98/5 \rfloor + \lfloor 98/25 \rfloor = 19 + 3 = 22$; sevens in $2024!$: $289 + 41 + 5 = 335$. Each floor counts one "layer": multiples of p, then the extra factor in multiples of p^2, and so on (Example 4).

📖 Worked examples

2016 AMC 8 #5 · translate to digits

Example 1. The number N is a two-digit number. When N is divided by 9, the remainder is 1. When N is divided by 10, the remainder is 3. What is the remainder when N is divided by 11?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 5 **(E)** 7

🧰 #1 Anatomy #3 Rule families

Key idea: each remainder fact is a digit fact. "Remainder 3 mod 10" says the units digit is 3; "remainder 1 mod 9" says the digit sum is 1 more than a multiple of 9. Scan $13, 23, 33, \ldots$: only 73 has digit sum 10 ✓. Then $73 = 11 \times 6 + 7$. **Answer: E**

2017 AMC 8 #7 · repeat-block

Example 2. Let Z be a 6-digit positive integer, such as 247247, whose first three digits are the same as its last three digits taken in the same order. Which of the following numbers must also be a factor of Z?

(A) 11 **(B)** 19 **(C)** 101 **(D)** 111 **(E)** 1111

🧰 #7 Repeat-block

Key idea: copy-pasting a 3-digit block is multiplication:

$$Z = \overline{abc} \times 1000 + \overline{abc} = \overline{abc} \times 1001 = \overline{abc} \times 7 \times 11 \times 13.$$

So 7, 11 and 13 all divide every such Z — and 11 is on the list. (The others aren't guaranteed: $247247/101$ isn't even an integer.) **Answer: A**

2020 AMC 8 #19 · split the divisor

Example 3. A number is called flippy if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by 15?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

🧰 #3 Rule families

Key idea: $15 = 3 \times 5$ — test both parts. A five-digit flippy number looks like $\overline{ababa}$. Divisible by $5 \Longrightarrow$ it ends in 0 or 5; the last digit is a, which is also the *leading* digit, so $a \neq 0$, forcing $a = 5$: the number is $\overline{5b5b5}$. Divisible by $3 \Longrightarrow$ digit sum $15 + 2b$ is a multiple of $3 \Longrightarrow b \in \{0, 3, 6, 9\}$ — four numbers ($50505, 53535, 56565, 59595$). **Answer: B**

2017 AMC 8 #19 · factorials

Example 4. For any positive integer M, the notation $M!$ denotes the product of the integers 1 through M. What is the largest integer n for which 5^n is a factor of the sum $98! + 99! + 100!$?

(A) 23 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

🧰 #10 Legendre

Key idea: factor out the smallest factorial first:

$$98! + 99! + 100! = 98!\,(1 + 99 + 99 \times 100) = 98! \times 10000 = 98! \times 2^4 \times 5^4.$$

Now count the fives inside $98!$ with floors: $\lfloor 98/5 \rfloor + \lfloor 98/25 \rfloor = 19 + 3 = 22$. Add the 5^4 from the 10000: $22 + 4 = 26$. **Answer: D**

✏️ Practice

Try each one before opening the solution.

2016 AMC 8 #20

1. The least common multiple of a and b is 12, and the least common multiple of b and c is 15. What is the least possible value of the least common multiple of a and c?

(A) 20 **(B)** 30 **(C)** 60 **(D)** 120 **(E)** 180

2017 AMC 8 #12

2. The smallest positive integer greater than 1 that leaves a remainder of 1 when divided by 4, 5, and 6 lies between which of the following pairs of numbers?

(A) 2 and 19 **(B)** 20 and 39 **(C)** 40 and 59 **(D)** 60 and 79 **(E)** 80 and 124

2018 AMC 8 #7

3. The 5-digit number $\underline{2}\,\underline{0}\,\underline{1}\,\underline{8}\,\underline{U}$ is divisible by 9. What is the remainder when this number is divided by 8?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 7 mock · course problem

4. What is the units digit of 13^{2012}?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

mock · course problem

5. How many three-digit numbers are divisible by 13?

(A) 7 **(B)** 67 **(C)** 69 **(D)** 76 **(E)** 77

2022 AMC 8 #17

6. If n is an even positive integer, the double factorial notation $n!!$ represents the product of all the even integers from 2 to n. For example, $8!! = 2 \cdot 4 \cdot 6 \cdot 8$. What is the units digit of the following sum?

$$2!! + 4!! + 6!! + \cdots + 2018!! + 2020!! + 2022!!$$

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

2016 AMC 8 #24

7. The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2019 AMC 8 #14

8. Isabella has 6 coupons that can be redeemed for free ice cream cones at Pete's Sweet Treats. In order to make the coupons last, she decides that she will redeem one every 10 days until she has used them all. She knows that Pete's is closed on Sundays, but as she circles the 6 dates on her calendar, she realizes that no circled date falls on a Sunday. On what day of the week does Isabella redeem her first coupon?

(A) Monday **(B)** Tuesday **(C)** Wednesday **(D)** Thursday **(E)** Friday

2018 AMC 8 #21

9. How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2020 AMC 8 #12

10. For a positive integer n, the factorial notation $n!$ represents the product of the integers from n to 1. (For example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.) What value of N satisfies the following equation?

$$5! \cdot 9! = 12 \cdot N!$$

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

🆕 Fresh from the latest exams

The same ideas, exactly as they appeared on the most recent tests.

2024 AMC 8 #1

11. What is the ones digit of

$$222{,}222 - 22{,}222 - 2{,}222 - 222 - 22 - 2\,?$$

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

2025 AMC 8 #6

12. Sekou writes down the numbers $15, 16, 17, 18, 19$. After he erases one of his numbers, the sum of the remaining four numbers is a multiple of 4. Which number did he erase?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

2026 AMC 8 #16

13. Consider all positive four-digit integers consisting of only even digits. What fraction of these integers are divisible by 4?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{3}{4}$

Sources: prep-course Lecture 17 (video + board notes, translated to English — this lecture has no homework-solutions chapter, so the worked examples were rebuilt from the board notes) and official AMC 8 problems from the local bank (`kb/content/banks/`). Answers verified against the bank's official answer keys. Companion handouts: Counting Divisors and Counting Multiples & Divisibility.

🗝️ Answer key

Example	1	2	3	4									
	E	A	B	D									
Practice	1	2	3	4	5	6	7	8	9	10	11	12	13
	A	D	B	A	C	B	A	C	E	A	B	C	D

📝 Solutions to practice problems

Practice 1.

🧰 #6 Build the LCM

b divides both $12 = 2^2 \cdot 3$ and $15 = 3 \cdot 5$, so b is 1 or 3 — b carries no 2s and no 5s. Then a must supply the whole 2^2 (so $4 \mid a$) and c must supply the 5 (so $5 \mid c$). Choosing $a = 4$, $b = 3$, $c = 5$ works, and $\operatorname{lcm}(4,5) = 20$. **Answer: A**

Practice 2.

🧰 #6 LCM shift

Same remainder 1 for all three divisors $\Longrightarrow N - 1$ is a common multiple of $4, 5, 6$. The smallest positive one is $\operatorname{lcm}(4,5,6) = 2^2 \cdot 3 \cdot 5 = 60$ (not $4 \cdot 5 \cdot 6 = 120$ — don't double-count the shared 2!). So $N = 61$, which lies between 60 and 79. **Answer: D**

Practice 3.

🧰 #3 Rule families

Digit-sum rule: $2 + 0 + 1 + 8 + U = 11 + U$ must be a multiple of 9, so $U = 7$. Last-three-digits rule for 8: $187 = 8 \times 23 + 3$ — remainder 3. **Answer: B**

Practice 4.

🧰 #5 Power cycles

Only the base's units digit matters: 13^{2012} ends like 3^{2012}. Powers of 3 end $3, 9, 7, 1, 3, 9, \ldots$ — cycle length 4. Since 2012 is a multiple of 4, we land on the *last* digit of the cycle: 1. **Answer: A**

Practice 5.

🧰 #8 First & last index

First three-digit multiple: $13 \times 8 = 104$. Last: $13 \times 76 = 988$ (since $13 \times 77 = 1001$). Count the indices: $76 - 8 + 1 = 69$. **Answer: C**

Practice 6.

🧰 #5 Units digits

From $10!!$ onward every term contains the factor 10, so those units digits are all 0. Only four terms matter: $2!! = 2$, $4!! = 8$, $6!! = 48$, $8!! = 384$ — units digits $2 + 8 + 8 + 4 = 22$, which ends in 2. **Answer: B**

Practice 7.

🧰 #3 Rule families

Divisible by $5 \implies S = 5$ (no digit 0 available). Divisible by $4 \implies$ the two-digit tail $\overline{QR}$ is a multiple of 4: from the digits 1–4 that's 12, 24 or 32. Divisible by $3 \implies R + S + T = R + 5 + T$ is a multiple of 3. Test: $QR = 12$ leaves $T \in \{3, 4\}$, sums $10, 11$ ✗; $QR = 32$ leaves $T \in \{1, 4\}$, sums $8, 11$ ✗; $QR = 24$ leaves $T = 3$: $4 + 5 + 3 = 12$ ✓. So $Q = 2, R = 4, T = 3$ and P is the leftover digit: 1. **Answer: A**

Practice 8.

🧰 #9 Mod 7 calendar

The six dates are $0, 10, 20, 30, 40, 50$ days after the first — mod 7 that's $0, 3, 6, 2, 5, 1$: six *different* weekday shifts, missing only 4. For no date to be a Sunday, Sunday must sit exactly 4 weekdays after the first coupon's day — and 4 days before Sunday is **Wednesday**. **Answer: C**

Practice 9.

🧰 #6 Same deficit

Spot the pattern: $2 = 6 - 4,\ \ 5 = 9 - 4,\ \ 7 = 11 - 4$ — every division falls short by the same 4. So $N + 4$ is divisible by 6, 9 and 11, i.e. by $\operatorname{lcm}(6, 9, 11) = 198$. Three-digit candidates: $N + 4 = 198, 396, 594, 792, 990$ — five values ($N = 194, \ldots, 986$). **Answer: E**

Practice 10.

🧰 #10 Factorials

$5! = 120 = 12 \times 10$, so $5! \cdot 9! = 12 \times 10 \times 9! = 12 \times 10!$ — matching $12 \cdot N!$ gives $N = 10$. **Answer: A**

Practice 11.

🧰 #5 Units digits

Group the subtractions: $22222 + 2222 + 222 + 22 + 2 = 24690$, and $222222 - 24690 = 197532$ — ones digit 2. (Units-only shortcut: $2 - (2 + 2 + 2 + 2 + 2) = 2 - 10 \equiv 2 \pmod{10}$, watching the borrows.) **Answer: B**

Practice 12.

🧰 #1 Anatomy #6 LCM shift

The five numbers add to 85. Erasing x must leave a multiple of 4: $85 - x \equiv 0 \pmod 4$, and $85 = 4 \times 21 + 1$, so $x \equiv 1 \pmod 4$ — among the choices only 17. Check: $85 - 17 = 68 = 4 \times 17$ ✓. **Answer: C**

Practice 13.

🧰 #3 Rule families

Divisibility by 4 only looks at the last two digits: $\overline{tu} = 10t + u \equiv 2t + u \pmod 4$. Here the tens digit t is *even*, so $2t$ is already a multiple of 4 — the condition collapses to $4 \mid u$, i.e. $u \in \{0, 4, 8\}$: 3 of the 5 possible even units digits. The first three digits never matter, so the fraction is $\frac{3}{5}$. **Answer: D**